Biostat 514
Emerson, Fall 09
Homework #4 Key
October 30, 2009

6. Simpsons Paradox suggests that for events $A$, $B$, and $C$, it is possible that
   - for a stratum with event $C$ being true, $Pr(A|BC) > Pr(A|B^cC)$
   - for a stratum with event $C$ not being true, $Pr(A|BC^c) > Pr(A|B^cC^c)$
   - but in the combined strata, $Pr(A|B) < Pr(A|B^c)$.

   We want to show that our definition of confounding is related to Simpsons Paradox. Thus, for events $A$, $B$, and $C$, with
   - $Pr(A|BC) > Pr(A|B^cC)$
   - $Pr(A|BC^c) > Pr(A|B^cC^c)$

   then show that either of the following conditions
   - $B$ and $C$ are independent, or
   - $A$ and $C$ are conditionally independent when conditioned on $B$ (so $Pr(AC|B) = Pr(A|B)Pr(C|B)$ )

   are sufficient (but not necessary) to guarantee $Pr(A|B) > Pr(A|B^c)$ thus avoiding Simpsons Paradox whenever we do not have confounding. (I want you to show both the sufficient and the not necessary parts.)

   **Answer:**

   First, we consider the case where $B$ and $C$ are independent.

   Now, if $B$ and $C$ are independent, then we can easily show that $B^c$ and $C$ are also independent, because:
   - By partitioning event $C$ into $C \cap B$ and $C \cap B^c$ and using the third axiom or probability (additivity of probabilities of mutually exclusive events), $Pr(C) = Pr(C \cap B) + Pr(C \cap B^c)$. This can be rearranged to provide $Pr(C \cap B^c) = Pr(C) - Pr(C \cap B)$.
   - Then, because $B$ and $C$ are independent, $Pr(C \cap B) = Pr(C)Pr(B)$.
   - Combining these two parts, we then find

     $$Pr(C \cap B^c) = Pr(C) - Pr(C)Pr(B) = Pr(C)\left(1 - Pr(B)\right) = Pr(C)Pr(B^c),$$

     thus by the definition of independent events, $C$ is independent of $B^c$.

   By direct analogy, events $B$ and $C^c$ are also independent, as are events $B^c$ and $C^c$.

Now, by the definition of conditional probabilities, for settings in which $0 < Pr(B) < 1$ and $0 < Pr(C) < 1$, we have

$$Pr(A|BC) = \frac{Pr(ABC)}{Pr(BC)} \quad Pr(A|B^cC) = \frac{Pr(AB^cC)}{Pr(B^cC)} \quad Pr(A|BC^c) = \frac{Pr(ABC^c)}{Pr(BC^c)} \quad Pr(A|B^cC^c) = \frac{Pr(AB^cC^c)}{Pr(B^cC^c)},$$

and using the assumed independence of $B$ and $C$, we have

$$Pr(A|BC) = \frac{Pr(ABC)}{Pr(B)Pr(C)} \quad Pr(A|B^cC) = \frac{Pr(AB^cC)}{Pr(B^c)Pr(C)}$$
$$Pr(A|BC^c) = \frac{Pr(ABC^c)}{Pr(B)Pr(C^c)} \quad Pr(A|B^cC^c) = \frac{Pr(AB^cC^c)}{Pr(B^c)Pr(C^c)}.$$

Then, from the assumed inequalities for the conditional probabilities within strata we have

$$\begin{aligned}
Pr(A|BC) > Pr(A|B^cC) &\Rightarrow \frac{Pr(ABC)}{Pr(B)Pr(C)} > \frac{Pr(AB^cC)}{Pr(B^c)Pr(C)} \\
&\Rightarrow \frac{Pr(ABC)}{Pr(B)} > \frac{Pr(AB^cC)}{Pr(B^c)} \\
&\Rightarrow Pr(AC|B) > Pr(AC|B^c), \text{ and} \\
Pr(A|BC^c) > Pr(A|B^cC^c) &\Rightarrow \frac{Pr(ABC^c)}{Pr(B)Pr(C^c)} > \frac{Pr(AB^cC^c)}{Pr(B^c)Pr(C^c)} \\
&\Rightarrow \frac{Pr(ABC^c)}{Pr(B)} > \frac{Pr(AB^cC^c)}{Pr(B^c)} \\
&\Rightarrow Pr(AC^c|B) > Pr(AC^c|B^c).
\end{aligned}$$

Thus by combining the larger probabilities with each other and the smaller probabilities with each other, we also know

$$Pr(AC|B) + Pr(AC^c|B) > Pr(AC|B^c) + Pr(AC^c|B^c).$$

Because conditional probabilities are probabilities, and because $A \cap C$ and $A \cap C^c$ form a partition of $A$, we know that

$$Pr(AC|B) + Pr(AC^c|B) = Pr(A|B) \qquad \text{and} \qquad Pr(AC|B^c) + Pr(AC^c|B^c) = Pr(A|B^c),$$

so if $B$ and $C$ are independent we must have

$$Pr(A|B) > Pr(A|B^c)$$

whenever $Pr(A|BC) > Pr(A|B^cC)$ and $Pr(A|BC^c) > Pr(A|B^cC^c)$.

Now, we assume that $A$ and $C$ are conditionally independent when conditioning on $B$, but we make no assumptions about the independence of $B$ and $C$.

First, I note that this problem would have been far easier if I had told you that $A$ and $C$ were also conditionally independent when conditioning on $B^c$. But I did not do that, and you cannot presume that such would be true. To see that, consider the equiprobable sample space $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Let $A = \{2, 3, 5, 7\}$ be the event that $\omega \in \Omega$ is prime, let $B = \{1, 2, 3, 4\}$ be the event that $\omega \in \Omega$ is less than 5, and let $C = \{2, 4, 6, 8\}$ be the event that $\omega \in \Omega$ is even. Then $A$ and $C$ are independent conditional on $B$, but $A$ and $C$ are mutually exclusive (and hence definitely not independent) conditional on $B^c$.

So for this problem we can only assume that $A$ and $C$ are independent conditional on $B$.

Now from the conditional independence we know that providing $Pr(BC)$, $Pr(BC^c)$, $Pr(B^cC)$, $Pr(B^cC^c)$ are all positive

$$\begin{aligned}
Pr(AC|B) = Pr(A|B)Pr(C|B) &\Rightarrow \frac{Pr(ABC)}{Pr(B)} = \frac{Pr(AB)}{Pr(B)}\frac{Pr(BC)}{Pr(B)} \\
&\Rightarrow Pr(ABC) = \frac{Pr(AB)Pr(BC)}{Pr(B)}, \text{ and} \\
Pr(AC^c|B) = Pr(A|B)Pr(C^c|B) &\Rightarrow \frac{Pr(ABC^c)}{Pr(B)} = \frac{Pr(AB)}{Pr(B)}\frac{Pr(BC^c)}{Pr(B)} \\
&\Rightarrow Pr(ABC^c) = \frac{Pr(AB)Pr(BC^c)}{Pr(B)}.
\end{aligned}$$

Substituting these expressions into the stipulated inequalities, we find

$$\begin{aligned} Pr(A|BC) > Pr(A|B^cC) &\Rightarrow \frac{Pr(ABC)}{Pr(BC)} > \frac{Pr(AB^cC)}{Pr(B^cC)} \\ &\Rightarrow \frac{Pr(AB)Pr(BC)}{Pr(B)Pr(BC)} > \frac{Pr(AB^cC)}{Pr(B^cC)} \\ &\Rightarrow Pr(B^cC)Pr(A|B) > Pr(AB^cC), \text{ and} \\ Pr(A|BC^c) > Pr(A|B^cC^c) &\Rightarrow \frac{Pr(ABC^c)}{Pr(BC^c)} > \frac{Pr(AB^cC^c)}{Pr(B^cC^c)} \\ &\Rightarrow \frac{Pr(AB)Pr(BC^c)}{Pr(B)Pr(BC^c)} > \frac{Pr(AB^cC^c)}{Pr(B^cC^c)} \\ &\Rightarrow Pr(B^cC^c)Pr(A|B) > Pr(AB^cC^c). \end{aligned}$$

Thus by combining the larger probabilities with each other and the smaller probabilities with each other, we also know

$$Pr(B^cC)Pr(A|B) + Pr(B^cC^c)Pr(A|B) > Pr(AB^cC) + Pr(AB^cC^c).$$

Now $B^c \cap C$ and $B^c \cap C^c$ form a partition of $B^c$, and $A \cap B^c \cap C$ and $A \cap B^c \cap C^c$ form a partition of $A \cap B^c$, so

$$\begin{aligned} &Pr(B^cC)Pr(A|B) + Pr(B^cC^c)Pr(A|B) > Pr(AB^cC) + Pr(AB^cC^c) \\ &\qquad \Rightarrow \left(Pr(B^cC) + Pr(B^cC^c)\right) Pr(A|B) > Pr(AB^cC) + Pr(AB^cC^c) \\ &\qquad \Rightarrow Pr(B^c)Pr(A|B) > Pr(AB^c) \\ &\qquad \Rightarrow Pr(A|B) > \frac{Pr(AB^c)}{Pr(B^c)} = Pr(A|B^c), \end{aligned}$$

which is what we were trying to show.

To show that it is not necessary to have either of those conditions in order to avoid Simpson's paradox, we need only find one example where $B$ and $C$ are not independent, $A$ and $C$ are not independent when conditioning on $B$ and when conditioning on $B^c$, $Pr(A|BC) > Pr(A|B^cC)$, $Pr(A|BC^c) > Pr(A|B^cC^c)$, and $Pr(A|B) > Pr(A|B^c)$.

We thus consider the equiprobable sample space $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

- Let $A = \{1, 2, 4, 6\}$, $B = \{1, 2, 3\}$, and $C = \{1, 4, 5\}$
- Then $Pr(A) = 0.5$; $Pr(B) = 0.375$; $Pr(C) = 0.375$
- $B \cap C = \{1\}$; $B \cap C^c = \{2, 3\}$; $B^c \cap C = \{4, 5\}$; $B^c \cap C^c = \{6, 7, 8\}$
- $A \cap C = \{1, 4\}$
- $A \cap B \cap C = \{1\}$; $A \cap B \cap C^c = \{2\}$; $A \cap B^c \cap C = \{4\}$; $A \cap B^c \cap C^c = \{6\}$
- $B$ and $C$ are not independent: $Pr(BC) = 0.125 \neq 0.375 \times 0.375$
- $A$ and $C$ are not independent conditionally on $B$: $Pr(AC|B) = 0.333 \neq Pr(A|B)Pr(C|B) = 0.667 \times 0.333$
- $A$ and $C$ are not independent conditionally on $B^c$: $Pr(AC|B^c) = 0.2 \neq Pr(A|B^c)Pr(C|B^c) = 0.4 \times 0.4$
- $Pr(A|BC) = 1.0 > Pr(A|B^cC) = 0.5$
- $Pr(A|BC^c) = 0.5 > Pr(A|B^cC^c) = 0.333$
- $Pr(A|B) = 0.667 > Pr(A|B^c) = 0.4$

The point of all the above is that when there is no confounding, we are guaranteed that Simpson's paradox cannot happen. But the presence of confounding does not demand that Simpson's paradox is present. That is, with confounding, we might see an estimated association in the combined sample that is in the opposite direction than the stratum specific associations. But there are times that the estimated association in the combined sample is in the same direction as are the stratum specific associations, but there is still confounding. In the latter case, quantification of the association (e.g., difference in mean outcomes) will be different for the combined sample and within strata.