Biost 517
Applied Biostatistics I

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Lecture 10:
Inference About Means:
One Sample

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Lecture Outline

- Point Estimates
- Confidence Intervals
- Hypothesis Tests

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Inference for Means

- Most common parameter used as a basis for statistical inference is the mean
  - Tends to reflect a wide variety of differences between distributions
    - E.g., extremely sensitive to changes in the tail of distributions
  - Statistical theory allow us to know the sampling distribution, and thus allows us to do inference

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Point Estimate

- Most often estimate population mean with sample mean
  - Always unbiased estimate for the true mean
    - Tends to true mean across replicate experiments
  - Always consistent estimate for the true mean
    - Tends to true mean as sample size increases
  - Often minimum variability
    - Especially when the distribution of measurements is approximately normal

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Approximate Sampling Distn

- Sample means are asymptotically normally distributed

Data $(X_1, X_2, \ldots X_n)$ are
independent,
identically distributed,
with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$

For large n: $\overline{X} \sim N\left(\text{mean } \mu, \text{var } \frac{\sigma^2}{n}\right)$

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What Should We Expect?

- If true mean is μ, true SD is σ, and sample size is n: sample mean should be

between			with probability
$\mu - 1.645\frac{\sigma}{\sqrt{n}}$	and	$\mu + 1.645\frac{\sigma}{\sqrt{n}}$	90%
$\mu - 1.96\frac{\sigma}{\sqrt{n}}$	and	$\mu + 1.96\frac{\sigma}{\sqrt{n}}$	95%
$\mu - 2.576\frac{\sigma}{\sqrt{n}}$	and	$\mu + 2.576\frac{\sigma}{\sqrt{n}}$	99%
$\mu - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$	and	$\mu + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$	$100(1-\alpha)\%$

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Confidence Intervals

- Frequentist confidence interval estimates
  - "For what values of the population parameter are these data fairly typical?"
  - How should we define "typical"?
    - Not in the upper extreme of the sampling distn?
    - Not in the lower extreme of the sampling distn?
    - In neither of the "tails" of the sampling distn?
  - How should we define extreme?
    - 5%, 2.5%, 1%?

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Lower Confidence Bound

- The true mean that is so low that we would not expect so high a sample mean
  - "not have expected" = probability is less than α

If $\mu = \mu_L$, with probability $100(1-\alpha)\%$ we expect

$$\overline{X} \le \mu_L + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$$

So $100(1-\alpha)\%$ lower confidence bound is

$$\mu_L = \overline{X} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$$

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Upper Confidence Bound

- The true mean that is so high that we would not expect so low a sample mean
  - “not have expected” = probability is less than α

If $\mu = \mu_U$, with probability $100(1-\alpha)\%$ we expect

$$\overline{X} \geq \mu_U - z_{1-\alpha} \frac{\sigma}{\sqrt{n}}$$

So $100(1-\alpha)\%$ upper confidence bound is

$$\mu_U = \overline{X} + z_{1-\alpha} \frac{\sigma}{\sqrt{n}}$$

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100(1-α)% Confidence Interval

- Set of all true means that reasonably result in the observed sample mean
  - “reasonably” = central 100(1-α)% of sampling distn

$100(1-\alpha)\%$ confidence interval is (μ_L, μ_U)

$$\mu_L = \overline{X} - z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$\mu_U = \overline{X} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$$

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Small Sample Adjustment

- Almost always have to estimate the population standard deviation
  - With continuous data, use the sample standard deviation s
- Use critical values for the t distribution
  - Exactly correct if the data were normal
  - Generally behaves well in other settings

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100(1-α)% Confidence Interval

- Set of all true means that reasonably result in the observed sample mean
  - “reasonably” = central 100(1-α)% of sampling distn

$100(1-\alpha)\%$ confidence interval is (μ_L, μ_U)

$$\mu_L = \overline{X} - t_{n-1,1-\alpha/2} \frac{s}{\sqrt{n}}$$

$$\mu_U = \overline{X} + t_{n-1,1-\alpha/2} \frac{s}{\sqrt{n}}$$

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t Distribution

- Properties of the t distribution
  - "Degrees of freedom" related to sample size
    - Usually the sample size minus the number of parameters used to estimate the mean
  - Symmetric about 0
  - Heavier tails with decreasing degrees of freedom
    - 1 degree of freedom is Cauchy (no mean)
    - Infinite degrees of freedom is standard normal

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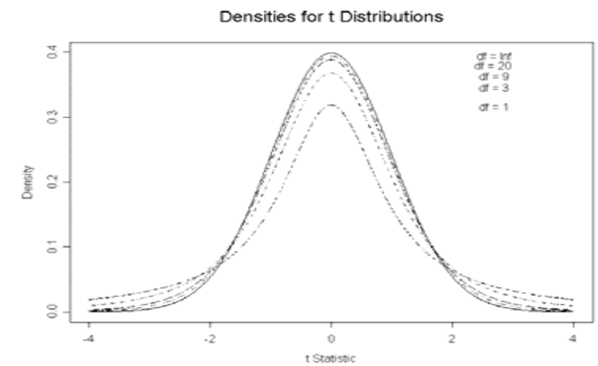
t Distribution Densities



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t Distribution Quantiles

- Selected upper quantiles of the t distribution: $t_{df,1-\alpha}$

df	.01	.025	.05
1	31.821	12.706	6.314
3	4.541	3.182	2.353
9	2.821	2.262	1.833
20	2.528	2.086	1.725
50	2.403	2.009	1.676
Inf	2.326	1.960	1.645

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Stata: CI for Population Mean

- "`ci` *var*, `level(#)`"
  - # is an integer between 10 and 99
  - a default can be set by "set level #"
- "`cii` *#n #mn #sd*, `level(#)`"
  - "immediate" CI by supplying n, sample mean, and sample standard deviation
- Commands for t test also give 95% CI
  - "`ttest` *var*"
  - "`ttesti` *#n #mn #sd #val*"

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Example: FEV in Smokers

```
. bysort smoker: ci fev

-> smoker = 0
Var | Obs  Mean  Std. Err.   [95% Conf. Interval]
fev | 589  2.57       0.04      2.50         2.63

-> smoker = 1
Var | Obs  Mean  Std. Err.   [95% Conf. Interval]
fev |  65  3.28       0.09      3.09         3.46
```

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Example: FEV in Smokers

- Best estimate of mean FEV
  - Nonsmokers: 2.57 l / sec
  - Smokers: 3.28 l / sec
- Interval estimate for mean FEV:
  - Nonsmokers: 95% confident that the true mean is between 2.50 l / sec and 2.63 l / sec
  - Smokers: 95% confident that the true mean is between 3.09 l / sec and 3.46 l / sec

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Interpretation of CI: Boring

- 95% CI for Smokers: 3.09 to 3.46 l / sec
- CORRECT, BUT BORING:
  - Of all CI computed in this manner, 95% of them will “cover” the true mean
    - (This says nothing about the numbers 3.09 and 3.46, because in repeated experiments, we would get different CI)

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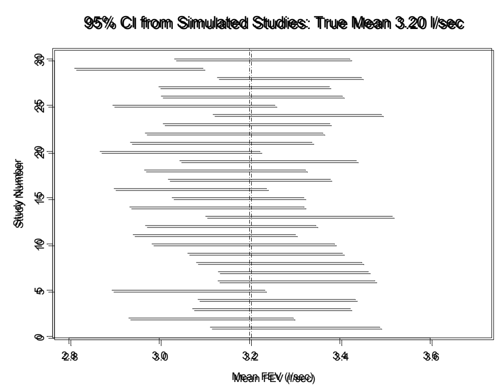
“Forest Plot” of Replicated CI



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Interpretation of CI: Wrong

- 95% CI for Smokers: 3.09 to 3.46 l / sec
- WRONG:
  - Probability is 95% that the true mean is between 3.09 and 3.46 l / sec
    - (This is a Bayesian statement that would have to be based on a prior distribution for the mean FEV)

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Interpretation of CI: Best

- 95% CI for Smokers: 3.09 to 3.46 l / sec
- BEST:
  - The sample mean we observed (3.28 l / sec) is typical of what we expect if the true mean were between 3.09 l / sec and 3.46 l / sec
    - (Frequentist confidence intervals are statements about the probability of observing data under hypothesized true values of the parameter)

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Comparison of CI

- 95% CI for Nonsmokers: 2.59 to 2.63 l/sec
- 95% CI for Smokers: 3.09 to 3.46 l/sec
  - The two CI do not overlap
    - There is no true value of the mean that would typically lead to both of these observations
    - With independent groups, this finding ensures a "statistically significant" difference between the true means
      - BUT: Statistical significance can occur with overlapping confidence intervals

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Hypothesis Tests of Means

- One sample t test: To test hypotheses that the true mean is in some particular range
  - Null Hypothesis
    - Usually status quo
      - We will tend to believe this unless we "prove" otherwise
    - What we hope to disprove
  - Alternative Hypothesis
    - What we hope is true

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One- vs Two-sided tests

- One sided test of greater alternative
  - Null H_0: $\mu \le \mu_0$ vs Alt : $\mu \ge \mu_1 > \mu_0$
- One sided test of lesser alternative
  - Null H_0: $\mu \ge \mu_0$ vs Alt : $\mu \le \mu_1 < \mu_0$
- Two sided test
  - Null H_0: $\mu = \mu_0$ vs Alt : $\mu \ne \mu_0$
    - Lower Alt: H_{1-}: $\mu < - \mu_1$
    - Null H_0 : $\mu = \mu_0$
    - Upper Alt: H_{1+}: $\mu > \mu_1$

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Choosing One- vs Two-sided

- What will change your behavior?
  - E.g., New treatment vs Placebo
    - One-sided test
      - Only adopt new treatment if better
  - E.g., Existing treatment vs Placebo
    - Two-sided test
      - Push new treatment if better
      - Neutral if about equal
      - Warn against new treatment if worse

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Computing P values

- Frequentist P value
  - Probability of observing as (or more) extreme results

Suppose we observe $\overline{X} = \overline{x}$

Lower one-sided P value $\Pr\left(\overline{X} \le \overline{x} \mid \mu = \mu_0\right)$

Upper one-sided P value $\Pr\left(\overline{X} \ge \overline{x} \mid \mu = \mu_0\right)$

Two-sided P value $\Pr\left(\left|\overline{X} - \mu\right| \le \left|\overline{x} - \mu_0\right| \mid \mu = \mu_0\right)$

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Computing P values using t

- Using t statistic

Standardized statistic $T = \dfrac{\overline{X} - \mu_0}{s / \sqrt{n}} \sim t_{n-1}$

Lower one-sided P value $\Pr\left(t_{n-1} \le \dfrac{\overline{x} - \mu_0}{s / \sqrt{n}}\right)$

Upper one-sided P value $\Pr\left(t_{n-1} \ge \dfrac{\overline{x} - \mu_0}{s / \sqrt{n}}\right)$

Two-sided P value $2 \times \Pr\left(t_{n-1} \ge \dfrac{\left|\overline{x} - \mu_0\right|}{s / \sqrt{n}}\right)$

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Stata: One sample t test

- Stata will give you everything you want quite easily
  - "`tprob (df, t)`"
    - Function returns two-sided P value
  - Performing t test:
  - "`ttest var = #val`"
  - "`ttesti #n  #mn   #sd   #val`"
    - provide P values for tests that the mean is equal to #val
    - provide 95% confidence intervals

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Example: PSA data set

- Test that mean nadir is 15.0 (Why?)

```
. ttest nadir=15
One-sample t test       Number of obs =       50

Varble | Mean StdErr    t   P>|t|  [95% CI]
-------+------------------------------------
 nadir | 16.36  5.55  2.95  0.005  5.21  27.51
Degrees of freedom: 49

Ho: mean(nadir) = 15
Ha: mean < 15    Ha: mean ~= 15     Ha: mean > 15
t =   0.2450     t =   0.2450       t =   0.2450
P<t=  0.5963    P>|t|= 0.8075      P>t=   0.4037
```

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Example: Interpretation

- Interpretation
  - The observed results are not atypical of those that might be obtained when the true mean is 15 (P=.81).
  - Based on these results, we cannot with 95% confidence reject the hypothesis that the true mean is 15.
  - We can reject with 95% confidence hypotheses that the true population mean is less than 5.21 or greater than 27.51.

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Extension to Other Settings

- The one sample tests of means are easily extended to other settings
  - Tests of geometric means
    - Perform inference on log transformed data
    - (Best to "back-transform" by exponentiating)
  - Tests for changes in paired samples
    - Perform inference on differences (or ratios) of measurements within individuals
  - Furthermore, the tests comparing two independent samples look much the same

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