Biost 517
Applied Biostatistics I

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Lecture 9:
Probability

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Lecture Outline

- Overview
- Properties of Probabilities
- Distributions of Random Variables
  – Binomial, Poisson, Exponential, Normal
- Inference Based on Summary Measures
- Sampling Distributions

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Use of Probability

- Recall that in statistical analysis we use probability in two ways
  – Probability models describe nondeterministic nature of measurements
    - Summary measures define “tend to”
  – Probability is used to quantify the uncertainty in the results (conclusions) of our statistical analysis

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Distributions of Statistics

- Probability theory relevant to quantifying uncertainty in our conclusions
  – Distributional theory for estimators and statistics across replicated experiments
    - Properties of expectation, variance, covariance
    - Asymptotic (large sample) results: Normal distribution
  – Criteria for “best” estimators and statistics
    - Bias (expectation), variance

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Special Distributions

- Common probability distributions used in probability models
  - Binomial, Poisson, Exponential, (Normal), etc.
- Probability distributions that are important for the sampling distribution of statistics
  - Normal
  - (Other sampling distns: t, F, chi squared)

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Properties of Probabilities

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Axioms of Probability

- Probabilities are between 0 and 1
  - $0 \leq Pr(A) \leq 1$
- Probability of the sample space is 1
  - $Pr(\Omega) = 1$ (something has to happen)
- Probabilities of disjoint events add
  - If events *A* and *B* cannot happen simultaneously, then

    $Pr(A \text{ or } B) = Pr(A) + Pr(B)$

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Properties of Probability

- Probability of the complement
  - $Pr(A^c) = 1 - Pr(A)$
- Probability of two simultaneous events
  - $Pr(A \text{ and } B) \leq Pr(A)$
- Probability of at least one event
  - $Pr(A) \leq Pr(A \text{ or } B) \leq Pr(A) + Pr(B)$
  - $Pr(A \text{ or } B) = Pr(A) + Pr(B) - Pr(A \text{ and } B)$
- Probability of a partition
  - $Pr(A) = Pr(A \text{ and } B) + Pr(A \text{ and } B^c)$

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Conditional Probability

- Probability of event *A* given *B* occurred
  - $Pr(A \mid B) = Pr(A \text{ and } B) / Pr(B)$
- Conditional probabilities are probabilities
  - $0 \le Pr(A \mid B) \le 1$
  - $Pr(B \mid B) = 1$
  - If $Pr(A_1 \text{ and } A_2 \text{ and } B) = 0$ (mut excl given *B*), then $Pr(A_1 \text{ or } A_2 \mid B) = Pr(A_1 \mid B) + Pr(A_2 \mid B)$
  - $Pr(A^c \mid B) = 1 - Pr(A \mid B)$

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Bayes' Rule

- Derived properties
  - $Pr(A \text{ and } B) = Pr(A \mid B) \times Pr(B)$
  - $Pr(A \text{ and } B) = Pr(B \mid A) \times Pr(A)$
  - $Pr(B) = Pr(B \mid A) \times Pr(A) + Pr(B \mid A^c) \times Pr(A^c)$
- Bayes' Rule

$$\Pr(A \mid B) = \frac{\Pr(A \text{ and } B)}{\Pr(B)} = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B \mid A)\Pr(A) + \Pr(B \mid A^c)\Pr(A^c)}$$

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Independent Events

- Definition: Events *A, B* are independent if
  - $Pr(A \text{ and } B) = Pr(A) \times Pr(B)$
- Properties
  - $Pr(A \mid B) = Pr(A)$
  - $Pr(B \mid A) = Pr(B)$
- NOTE: "Independence" and "mutually exclusive" are in some sense opposites

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Distributions of Random Variables

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Random Variables

- The idea of making some measurement
- Measurement value can vary
- Probability distribution describes the relative likelihood of observing particular values
- Once we know the probability distribution, we know everything

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Distribution Functions

- Specification depends on type of variable
- Categorical variables
  - List the probability for each possible value
    - A formula available for binomial, poisson
  - If ordered, can give cumulative probabilities (cdf)
- Continuous variables
  - Density or cumulative distribution function
    - Formulas available for density and/of cdf
      - Sometimes one is easier than the other

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Bernoulli Distribution

- A binary (0-1) random variable
  - E.g., sex, vital status
  - Probability distribution involves only the parameter *p:* $0 \leq p \leq 1$
    - $Pr(X = 1) = p$
    - $Pr(X = 0) = 1-p$
  - Mean $E(X) = p$
  - Variance $Var(X) = p(1-p)$
- A binary random variable ***must*** have a Bernoulli distribution

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Binomial Distribution

- Counts the number of events in *n* independent trials of the same experiment
  - The sum of *n* independent Bernoulli variables
  - Probability distribution has parameters *n,p*
    - For $k = 0, 1, 2, \ldots, n$

$$\Pr(X = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

  - Mean $E(X) = np$
  - Variance $Var(X) = np(1-p)$

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Poisson Distribution

- Counts the events occurring at a constant rate λ in a specified time (and space) *t*
  - Independent intervals of time and space
  - E.g., Heart attacks in Seattle in 2005
  - Probability distribution has parameter $\lambda > 0$
    - For $k = 0, 1, 2, 3, 4, \ldots$

$$\Pr(X = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

  - Mean $E(X) = \lambda t$
  - Variance $Var(X) = \lambda t$
- Poisson approx to Binomial for low *p*

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Exponential Distribution

- Continuous positive random variable having constant hazard
  - Sometimes used in time-to-event
    - "Memorylessness" precludes wide application
  - Probability distribution has hazard $\lambda > 0$

$$\Pr(X \le t) = 1 - e^{-\lambda t}$$

  - Mean $E(X) = 1/\lambda$
  - Variance $Var(X) = 1/\lambda^2$
- Poisson interarrival times are Exponential

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Normal (Gaussian) Distribution

- The "bell-shaped curve"
- Arises from sums of multiple effects
  - (due to the central limit theorem (CLT))
- Density and cdf from

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

  - Mean $E(X) = \mu$
  - Variance $Var(X) = \sigma^2$

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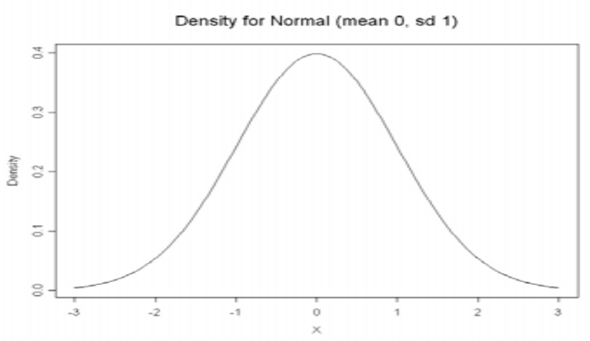
Normal Density



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Linearity of Normal Distribution

- If $X \sim N$ (mean μ, var σ^2), then for any constants *a, b*
  - Transformation $Y = aX + b$ is also normally distributed with
    - Mean $E(Y) = a\mu + b$
    - Variance $Var(Y) = a^2\sigma^2$

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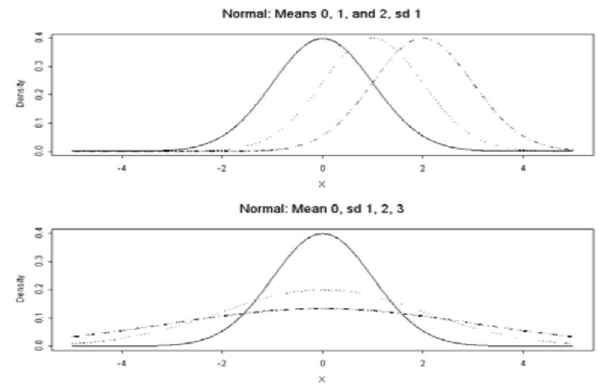
Varying Mean, Std Deviations



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Standardizing Transformation

- Important special linear transformation
  - $Z = (X - \mu)/\sigma \sim N(0,1)$
    - (and Z is very common notation here)
  - Allows computation of probabilities
- The density for the normal distribution cannot be integrated in closed form, instead use
  - numerical integration,
  - tables in the back of <u>any</u> statistics text tabulate probabilities for $Pr(Z \le c)$ (or equivalent)
  - statistical software

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Stata Commands

- "`normal`"
  - "`norm (c)`" gives $Pr(Z \le c)$ for $Z \sim N(0, 1)$
- Examples: *Z* normal with mean 0, std dev 1
  - Pr ($Z \le 1.07$): "`display norm (1.07)`"
  - Pr ($Z \ge 0.37$): "`display 1 - norm (0.37)`"
  - Pr ($0.37 \le Z \le 1.07$):
    - "`disp norm (1.07) – norm (0.37)`"

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Stata Commands

- For other normal distributions: Standardize
- Examples: *X* normal with mean 3, std dev 2
  - Z = (X – 3) / 2 is standard normal
  - Pr (X $\leq$ 1.07):
    - "`disp norm ( (1.07 - 3) / 2 )`"
  - Pr (X $\geq$ 0.37):
    - "`disp 1 - norm ( (0.37 – 3) / 2 )`"
- Examples: *X* normal with mean -4, std dev 5
  - Z= (X – -4) / 5 = (X + 4) / 5 is standard normal
  - Pr (X $\geq$ 1.07):
    - "`disp 1 - norm ( (1.07 + 4) / 5 )`"

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Sums of Independent Variables

- Sums of <u>independent</u> variables (no matter the distribution)
  - For independent random variables
    - *X* ~ (mean μ, var σ^2), and
    - *Y* ~ (mean τ, var θ^2)
  - The sum *X* + *Y* has moments (mean and variance)
    - mean = $\mu + \tau$, and
    - variance = $\sigma^2 + \theta^2$

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Sums of Normal Variables

- Sums of <u>independent</u> normally distributed variables are also normally distributed
  - For independent random variables
    - *X* ~ N (mean μ, var σ^2), and
    - *Y* ~ N (mean τ, var θ^2)
  - The sum *X* + *Y* is *normally distributed* with
    - mean = $\mu + \tau$, and
    - variance = $\sigma^2 + \theta^2$

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Inference Based on Summary Measures

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Real World

- Most often, we do not know the probability distribution for a random variable
- Scientific questions correspond to answering questions about unknown distributions
  - Prediction of individual measurements
  - Typical measurements, spread, etc.
  - Comparison of probability distributions across groups
    - Tendencies to be larger or smaller
    - Tendencies toward greater spread
    - Etc.

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Parameters of Distributions

- Generally we must define what we mean by “typical”, “spread”, etc.
- Usually we use some summary measure of the population’s data in a manner analogous to using descriptive statistics for a sample
  - Mean, median, mode
  - Proportion above some threshold
  - Variance
- Summary measures of population distributions are often called “parameters”

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Defining “tends to”

- Tendencies toward “larger” and “smaller” measurements based on comparing parameters of the distribution
- E.g., Scientific question:
  - Does the population of smokers tend to have smaller life expectancies than nonsmokers?
- Corresponding statistical question:
  - Is the median (or mean, etc.) life expectancy smaller for smokers compared to nonsmokers?

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Justification

- Do statistical questions about parameters really answer the scientific question: Yes and No
  - Certainly if parameters differ across populations, then the distributions differ
  - But: Two populations may have different distributions, but the same mean (median, …)
    - We need to be sure that a parameter used for inference captures what we care about

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Expectation of Variables

- Expectation of a random variable is simply its average value over the population
  - "Average" = "Mean" = "Expected Value"
    - For random variable *X*, this is written *E (X)*
  - For the same reasons that we used sample means to describe a sample, the population mean is often used to
    - Describe "typical" values in the population
    - Compare distributions across populations

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Other Population Parameters

- Median: *Mdn (X)* is just the value such that half the population is above it and half below it
  - More generally we can describe any quantile (percentile) for the distribution of *X* in the population
- Proportion of the population exceeding some threshold
- Variance: *Var (X)* is just the variance of the measurement across the entire population

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Scientific Uses of the Mean

- Allows prediction of totals for population
  - E.g., average health care costs can be used to predict total costs for larger populations
- Often sensitive to a wide variety of differences in distributions
  - Sensitive to outliers as well as shifts of the entire distribution

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Statistical Uses of the Mean

- Estimates of the mean are easy to use
- From mathematical theory we know
  - How to estimate the mean in an unbiased and/or consistent manner
  - How to quantify the precision of the estimates
- Easy to compute and combine across studies
  - Often the most efficient parameter estimated
    - Bernoulli, binomial, Poisson, exponential, normal

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Scientific Uses of the Variance

- Some scientific questions are related to the variability of the data
  - Quality control

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Statistical Uses of the Variance

- Because we often try to make inference about the population mean, we also tend to be interested in the population variance
  - The population variance can often be used to calculate the precision of our inference about the population mean

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Sampling Distributions

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Sampling Distributions

- We often quantify our confidence in study conclusions according to what would happen in replications of the experiment

- The distribution of a statistic across such replications is called the “sampling distribution” of the statistic

- E.g., Having observed 20% of smokers dying within a year, we might ask the probability of observing other results if we repeated the experiment many times

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Statistical Importance

- Both “frequentist” and “Bayesian” inference need sampling distributions
- Frequentist inference:
  - How often does data like this (or more extreme) happen under a particular hypothesis?
    - Purely a question about the sampling distribution
- Bayesian inference:
  - Given our data, what is the probability that a particular hypothesis is true?
    - Uses sampling distribution along with “prior distribution” (prevalence) in Bayes’ rule

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Bias and Variability of Statistics

- Often the exact sampling distribution for an estimator or test statistic is unknown or too difficult to compute
- Theory about expectations allows us to describe general tendencies of statistics
  - Unbiasedness: If we repeated the study many times, would the resulting statistics tend to be centered on the true parameter?
  - Variability: How variable would the estimates be across replicated experiments?

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Using Properties of Expectation

- We sometimes want to combine statistics reported in the literature
  - Perform *ad hoc* adjustment for covariates
  - Perform *ad hoc* meta-analyses across studies
- The basic properties of expectation often allow us to perform such informal analyses

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Linearity of Expectation

- For any
  - random variables *X, Y,* and
  - constants *a, b, c*

$$E(aX + bY + c) = aE(X) + bE(Y) + c$$

- (There is no requirement of independence or identical distribution)

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Sample Mean is Unbiased

- Across replicated studies, the average sample mean will tend to be the true mean
  - Population in which the average value is μ
  - Many studies each with n measurements
    - In each study calculate the sample mean
  - Average of those sample means tend to μ

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Properties of Variances

- The properties of expectation also help us describe the variability of statistics
- Recall that variance is the average squared distance from the mean
  - The fact that it is an average allows us to use the properties of expectation
  - The fact that it involves squaring the data means that we have to be careful as we use those properties

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Variance: Linear Transformations

- For any
  - random variable X, and
  - constants a, c

$$Var(aX + c) = a^2 Var(X)$$

  - (Recall that the variance involves subtracting off the mean and then squaring)

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Variance of Sums of Variables

- Variance of sums (and differences) of random variables
  - For any random variables *X, Y* with
    - *Corr (X, Y)* = ρ (the Greek letter rho)

$$Var(X + Y) = Var(X) + Var(Y) + 2\rho\sqrt{Var(X)\,Var(Y)}$$
$$Var(X - Y) = Var(X) + Var(Y) - 2\rho\sqrt{Var(X)\,Var(Y)}$$

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Sums of Independent Variables

- Variance of sums (and differences) of independent random variables
  - For any independent random variables *X, Y*
    - (so *Corr (X, Y)* = 0)

$$Var(X+Y) = Var(X) + Var(Y)$$
$$Var(X-Y) = Var(X) + Var(Y)$$

  - (recall that multiplying Y by -1 would multiply the variance by the square of -1)

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Variability of Sample Means

- Computed across replicate experiments
  - Suppose we sample *n* independent measurements from a population in which
    - the average value in the population is μ, and
    - the variance in the population is σ^2
  - Across a large number of replicated studies
    - the average sample mean will be μ, and
    - the variance of the sample means will be σ^2 / n

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Correlated Observations

- If the measurements are not independent of one another, then the preceding formula does not hold
  - Positively correlated measurements lead to greater variability of a group's sample means
    - Repeated measurements made on the same individual, family, etc. tend to be positively correlated
  - Negatively correlated measurements lead to less variable sample means for a group

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Standard Errors of Statistics

- Recall that we usually find it easier to use the square root of variances
  - Variance is in squared units
- "Standard errors" are the square root of the variance of a statistic
  - The major motivation for the nomenclature is to distinguish it from the standard deviation in the population of measurements

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Properties of Standard Errors

- Must be derived from variances
- We compute standard errors by first finding the sampling variance of the statistic
- When transforming or combining statistics
  - Convert standard errors to variances by squaring
  - Use properties of variances
    - (These properties are derived from expectations)
  - Convert resulting variance back to a standard error

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Using Standard Errors

- Knowing the bias and standard error of statistics to estimate sampling distribution
  - From properties of standard deviations
    - At least 89% of a population must be within 3 standard deviations of its mean (Lec 4, slide 20)
    - Thus, for any given statistic, we would expect the statistic to be within 3 standard errors of its mean in at least 89% of all studies
- (We would, however, prefer more precise information about the sampling distribution)

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Asymptotic Theory

- When the exact sampling distribution is unknown or too difficult to compute
  - Theoretical results allow statistical inference in the setting of sufficiently large sample sizes
  - Most of this asymptotic theory is based on some form of a central limit theorem for the distribution of a sum or arithmetic mean
    - "Asymptotics" = as the sample size becomes infinite

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Central Limit Theorem

- Sample means are asymptotically normally distributed

Data $(X_1, X_2, \ldots X_n)$ are
independent,
identically distributed,
with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$

For large n: $\bar{X} \dot{\sim} N\left(\text{mean } \mu, \text{var } \frac{\sigma^2}{n}\right)$

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Ex: Binomial Distribution

- Recall that Binomial random variables can be viewed as the sum of *n* independent, identically distributed Bernoulli variables

Independent Bernoulli random variables

$$Y_1, \dots, Y_n \text{ with } \begin{cases} \Pr(Y_i = 1) = p \\ \Pr(Y_i = 0) = 1 - p \end{cases}$$

$$X = (Y_1 + \cdots + Y_n) \sim Binomial(n, p)$$

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Ex: Binomial Distribution

- Application of CLT: Normal approximation to binomial

Central limit theorem : For large n,

$$\frac{X}{n} = \frac{1}{n}(Y_1 + \cdots + Y_n) = \overline{Y} \,\dot{\sim}\, N\left(p, \quad \frac{p(1-p)}{n} \right)$$

Properties of normal :

$$X \,\dot{\sim}\, N(\, np, \; np(1-p) \,)$$

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Ex: How Large is Large?

- "Rules of Thumb" for the normal approximation to the binomial:
  - Based on ensuring not many "outliers"
    - $np > 5$ and $n(1-p) > 5$; OR
    - $np(1-p) > 5$

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Limitations of CLT

- The definition of "large" sample size will be dependent upon the original distribution
  - Distributions with heavy tails require larger sample sizes for a good approximation
- That having been said:
  - It is often surprising how small "large" is
    - A sample size of 30 – 40 will suffice for most data sets commonly encountered in practice
    - (see Lumley, et al., *Annual Rev. Pub Health,* 2002)

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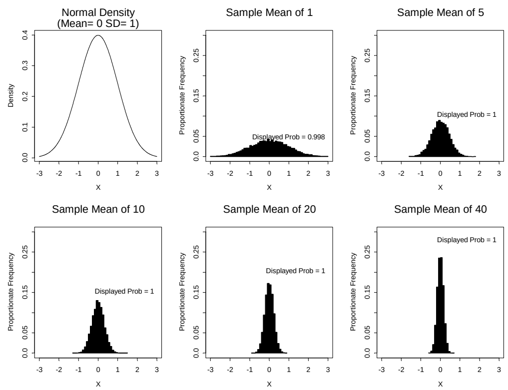
Normal Data



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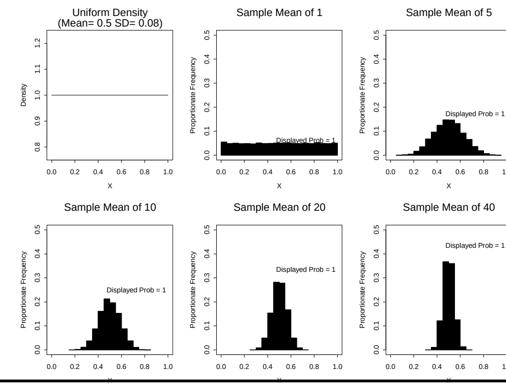
Uniform Data



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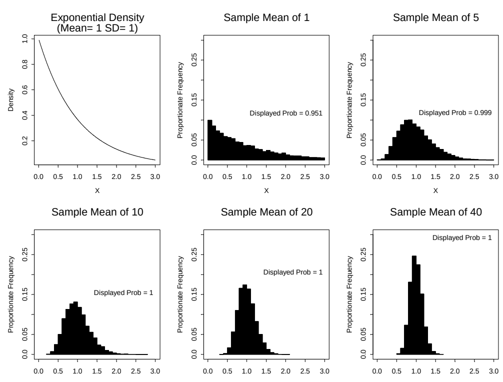
Exponential Data



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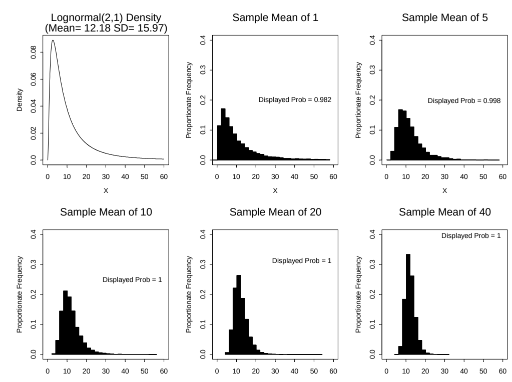
Lognormal Data (Mod. Skewed)



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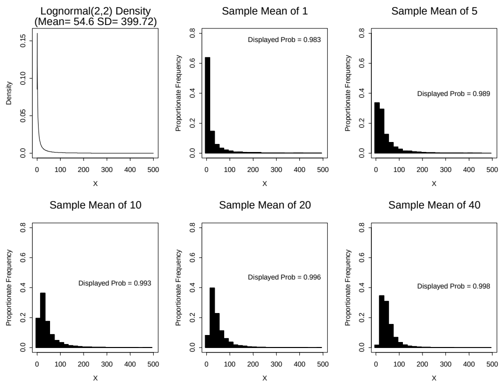
Lognormal Data (High Skewed)



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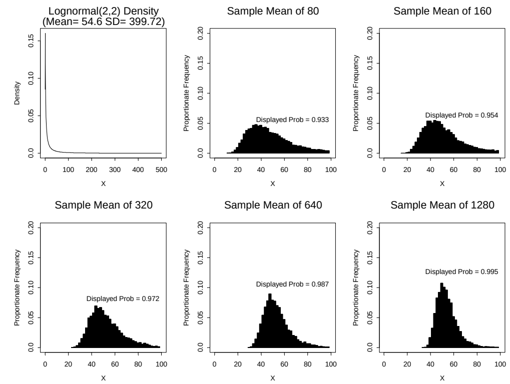
Lognormal Data (High Skewed)



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Other Central Limit Theorems

- CLTs exist in some other settings
  - Independent, but not identically distributed
  - Correlated observations
  - Transformations of sample means
- We will generally leave it to those who write computer programs to get the formulas right
  - But we must be aware of the additional conditions beyond finite variance
  - And we will have to tell the computer programs of the need to use the special formulas

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