Biost/Stat 533
Emerson, Spr 14
Homework #5
May 16, 2014

Written problems to be handed in Wednesday, May 21.

Under no circumstances may you refer to a homework key from this or other classes. While you may work with other students to derive a solution, when you write up your final solution, you may not refer to any other source. You must be able to develop your answer as if it were being done in a closed book, closed notes examination. You must provide a signed pledge to that effect:

On my honor I have neither given nor received unauthorized aid on the completion of this homework.

1. Consider a linear regression model relating response n-vector $\vec{Y}$ to an intercept and one predictor vector $\vec{X}$ parameterized as
$$\vec{Y} = \alpha + \vec{X}\beta + \vec{\epsilon},$$
with $\vec{\epsilon} \sim (\vec{0}, \sigma^2 \mathbf{I}_n)$. Define pairwise slope estimates for $i \neq j$ as
$$\hat{\beta}_{(i,j)} = \begin{cases} \frac{Y_i - Y_j}{X_i - X_j} & \text{if } X_i \neq X_j, \\ c & \text{else} \end{cases}$$
(where the choice of c in the second case is entirely arbitrary). Explicitly show that OLSE $\hat{\beta}$ satisfies
$$\hat{\beta} = \sum_{i=1}^{n} \sum_{j \neq i} w_{ij} \, \hat{\beta}_{(i,j)}$$
where the weights are related to the inverse variances of $\hat{\beta}_{(i,j)}$.

2. Random variable $X_1 \sim F_1$ is said to be stochastically larger than random variable $X_2 \sim F_2$ if $F_2(y) \geq F_1(y)$ for all $y \in (-\infty, \infty)$ (i.e, X_1 has a greater probability of being greater than y than does X_2).
   a. Show that for some $n \geq 1$ and some $\delta_1 > \delta_2 \geq 0$, non-central chi squared random variable $X_1 \sim \chi^2_n(\delta_1)$ is stochastically larger than $X_2 \sim \chi^2_n(\delta_2)$.
   b. Show that for some $n_1 > n_2 \geq 1$ and some $\delta \geq 0$, non-central chi squared random variable $X_1 \sim \chi^2_{n_1}(\delta)$ is stochastically larger than $X_2 \sim \chi^2_{n_2}(\delta)$.
   c. Show that when we consider the regression model $\vec{Y} = \mathbf{X}\vec{\beta} + \vec{\epsilon}$ with $\vec{\epsilon} \sim, (\vec{0}, \sigma^2 \mathbf{I}_n)$, the results in parts a and b demonstrate that the hypothesis test of $H_0 : \mathbf{A}\vec{\beta} = \vec{\mathbf{c}}$ based on the OLSE $\widehat{\vec{\beta}}$ and the quadratic form
$$Q = \frac{(\mathbf{A}\widehat{\vec{\beta}} - \vec{c})^T \, (\mathbf{A}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{A}^T)^{-1} \, (\mathbf{A}\widehat{\vec{\beta}} - \vec{c})}{\sigma^2}$$
is asymptotically unbiased and consistent.