Stat 512
Emerson, Fall 15
Final Examination
December 14, 2015

Instructions:

- **This exam is closed book, closed notes. No use of calculators is permitted.**
- **Write answers to the following questions on separate sheets of paper, starting each problem at the top of a new page. Use only the front side of each page. Be sure to write your name on the top of each page.**
- **In order to receive full credit, you must make clear how you derived the answers to the problems.**
- **You are allowed 1 hour and 40 minutes for this exam. When time is called, you must put down your pencils**

In addition to the theorems covered in class, you may use the following facts without proof:

· The following sums

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

If $X \sim \mathcal{E}(\lambda)$ (an exponential random variable with density $f_X(x) = \lambda e^{-\lambda x}\mathbf{1}_{[0<x<\infty]}$ for some $\lambda > 0$), then $EX = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$.

· As $n \to \infty$

$$\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^n = e^a$$

1. (15 points) Provide definitions for the following terms. Be sure to make clear any notation you use.

   a. Convergence almost surely.

   b. Convergence in probability.

   c. Convergence in distribution.

   d. The delta method.

   e. Slutsky's theorem.

2. (15 points) State and prove a weak law of large numbers.

3. (30 points) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. discrete random variables having probability mass function

$$p_X(x) = \frac{2x}{\theta(\theta+1)} \mathbf{1}_{x \in \{1,2,3,\ldots,\theta\}}$$

for some integer $\theta \geq 1$.

   a. Derive a method of moments estimator for θ for sample size n. Find its bias function and show its consistency.

   b. Find the asymptotic distribution for your estimator in part a.

   c. Find a maximum likelihood estimator for θ for sample size n.

4. (40 points) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. exponential random variables having the density $f_X(x) = \lambda e^{-\lambda x} \mathbf{1}_{[0<x<\infty]}$.

   a. Find a method of moments estimator for λ and give its asymptotic distribution.

   b. Find a maximum likelihood estimator for λ and give its asymptotic distribution.

   c. Find a maximum likelihood estimator for $\theta = Pr[X_1 > 1]$ and give its asymptotic distribution.

   d. Let $Y_i = \mathbf{1}_{[X_i>1]}$. Find a maximum likelihodd estimator for θ based on the Y_i's and give its asymptotic distribution.

   e. Discuss the relative advantages and disadvantages of your estimators in parts c and d.

5. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables having mean μ and variance σ^2, and let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables having mean ν and variance τ^2. Further suppose that X_i and Y_i are sampled in pairs, with X_i and Y_j are independent if $i \neq j$, but that the correlation between X_i and Y_j is ρ if $i = j$.

   a. (10 points) Find a method of moments estimator for $\mu + \nu$ and derive its asymptotic distribution. Is your estimator unbiased? Consistent?

   b. (10 points) Define $W_i = X_i^2 + Y_i^2$. Let $\theta = E[W_i]$. Find an unbiased estimator of θ. Is your estimator consistent?

   c. (10 points) Show that the estimator $\tilde{\theta} = \overline{X}_n^2 + \overline{Y}_n^2$ is a biased estimator for θ. Find the bias function.

   d. (10 points) Show that $\tilde{\theta}$ is not consistent for θ.

   e. (Bonus: 20 points) Find a method of moments estimator for $\mu^2 + \nu^2$ and derive its asymptotic distribution.