Stat 512
Emerson, Fall 15
Homework #2
October 7 2015

Written solutions to the homework problems are due on Wednesday October 14, 2015 at the beginning of class.

The homework problems are divided into "regular" and "more involved" problems. In order to facilitate multiple graders, you should hand in these categories of problems separately. That is, hand in one paper that contains only the "regular" problems, and another paper that contains only the "more involved" problems.
As noted on the syllabus and discussed during the first class, copying of homework solutions is not allowed and, when detected, will be investigated as an infraction of the academy integrity policy of the University of Washington. While it is permissible to discuss problems with other students, TAs, or the instructor in order to learn how to solve a problem, your written solutions must be prepared without directly referencing any notes or solutions derived from other students or sources found on the internet.

REGULAR PROBLEMS

1. Suppose continuous random variable X belongs to the family of all distributions having a linear probability density function (pdf) over the interval $[0, 1]$ and zero elsewhere. Let θ be the slope of the pdf.
   (a) Derive the formula for the pdf $f(x)$ of this family, making clear the range of permissible values of θ.
   (b) Derive the formula for the cumulative distribution function (cdf) $F(x)$ for this family.
   (c) Derive an expression for the median value of X as a function of θ.

2. Now consider that continuous random variable X belongs to a two parameter family of all distributions having a linear probability density function (pdf) with slope θ over open intervals $(0, \eta)$.
   (a) Derive the formula for the pdf $f(x)$ of this family, making clear the range of permissible values of θ and η.
   (b) Derive the formula for the cumulative distribution function (cdf) $F(x)$ for this family.
   (c) Derive an expression for the median value of X as a function of θ and η.

3. Suppose random variable X has a negative binomial distribution for some $p \in [0,1]$ and some integer $r \geq 1$ with

$$Pr[X = x] = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r} & x \in \{r, r+1, r+2, \dots\} \\ 0 & else \end{cases}$$

Prove that the above formula is a probability mass function.

4. Suppose discrete random variable X belongs to the family of all distributions having probability mass function (pmf) of the form

$$p(x) = Pr[X = x] = \begin{cases} \frac{c(\theta)}{\theta^x} & x \in \{0, 1, 2, \dots\} \\ 0 & else \end{cases}$$

State clearly the range of permissible values of θ and explicitly providing the form of $c(\theta)$. What is the name of this parametric family?

5. Suppose continuous random variable X has the exponential distribution $X \sim \mathcal{E}(\lambda)$ with pdf $f(x) = \lambda e^{-\lambda x} \mathbf{1}_{(0,\infty)}(x)$ for $\lambda > 0$.

   (a) What is the cumulative distribution function (cdf) for X?

   (b) What is the median for X?

   (c) For arbitrary positive s and t, find an expression for $Pr(X > s + t \,|\, X > s)$.

   (d) Suppose the distribution of X describes the time until failure of a light bulb that is left on continuously. Suppose it is still burning at time s. What is the "median residual lifetime" of the bulb after time s (i.e., the time after s that there is exactly a 50% chance the bulb will still be lit).

6. Suppose continuous random variables X has the standard uniform distribution $X \sim \mathcal{U}(0,1)$ with pdf $f(x) = \mathbf{1}_{(0,1)}(x)$. What are the cumulative distribution function and probability density function for $W = \log(X)$

MORE INVOLVED PROBLEMS

7. Suppose independent continuous random variables X_1 and X_2 have the exponential distribution $X \sim \mathcal{E}(\lambda)$ with pdf $f(x) = \lambda e^{-\lambda x} \mathbf{1}_{(0,\infty)}(x)$ for $\lambda > 0$.

   (a) What are the cumulative distribution function and probability density function for $W = \min(X_1, X_2)$?

   (b) What are the cumulative distribution function and probability density function for $W = \max(X_1, X_2)$?

8. Let X_1 and X_2 be Poisson random variables with $X_i \sim \mathcal{P}(\lambda_i)$ and probability mass functions

$$p_i(k) = Pr[X_i = x] = \begin{cases} \frac{e^{-\lambda_i}\lambda_i^x}{x!} & x \in \{0, 1, 2, \dots\} \\ 0 & else \end{cases}$$

Further suppose that X_1 and X_2 are independent random variables, so that the events $\{X_1 = x_1\}$ and $\{X_2 = x_2\}$ are independent for all $x_1, x_2 \in \{0, 1, 2, \dots\}$. *(Note that we will eventually describe methods for deriving these answers using convolutions. However, this problem can be answered by just considering independent events and the results we have discussed for conditional probabilities.)*

(a) What is the joint probability mass function $p_{X_1,X_2}(k_1, k_2) = Pr(X_1 = k_1 \cap X_2 = k_2)$.

(b) Letting random variable $S = X_1 + X_2$, what is the probability mass function $p_S(s)$.

(c) Find the conditional probability mass function $p_{X_1|S}(x|S = s)$. To what parametric family does this family belong.

9. Consider an experiment consisting of 2 independent rolls of a weighted die. Let X_k be the random variable measuring the number showing on top of the die on the kth roll. Hence, the sample space for X_k is $\Omega_X = \{1, 2, 3, 4, 5, 6\}$, with $P_X(X_k = \omega) = p_\omega$ for all $\omega \in \Omega_X$ and for $k = 1, 2$.

Define $S = X_1 + X_2$.

(a) What is the support of S?

(b) Can you find values for the parameters $p_1, \dots, p_6$ such that all possible outcomes for S are equally likely? If so, do so. If not, prove that it cannot be done.