Stat 512
Homework key 2

October 4, 2015

REGULAR PROBLEMS

1. Suppose continuous random variable X belongs to the family of all distributions having a linear probability density function (pdf) over the interval $[0,1]$ and zero elsewhere. Let θ be the slope of the pdf.

   (a) Derive the formula for the pdf $f(x)$ of this family, making clear the range of permissible values of θ.

   (b) Derive the formula for the cumulative distribution function (cdf) $F(x)$ for this family.

   (c) Derive an expression for the median value of X as a function of θ.

   Ans:

   (a) Suppose

   $$f(x) = \begin{cases} \theta x + b & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   where $b \in \mathbb{R}$ whose value may depend on θ. Integrating, we get that in the interval $[0,1]$, $F(x) = \theta \frac{x^2}{2} + bx + c$ where $c \in \mathbb{R}$. Now since $F(0) = 0$, $c = 0$. Therefore, on this interval, the cdf

   $$F(x) = \theta \frac{x^2}{2} + bx. \tag{2}$$

   Now,

   $$\begin{aligned} F(1) &= 1 \\ \Rightarrow b &= 1 - \frac{\theta}{2} \end{aligned}$$

   Now we need to find the range of θ. The only condition that need to be satisfied is $f(x) \geq 0$. Let $\theta \geq 0$. Then

   $$\begin{aligned} \min_{[0,1]} f(x) &= \min_{[0,1]} \theta x + b = b \geq 0. \\ \Rightarrow \theta &\leq 2. \end{aligned}$$

   Let $\theta \leq 0$. Then

   $$\begin{aligned} \min_{[0,1]} f(x) &= \min_{[0,1]} \theta x + b = \theta + b \geq 0. \\ \Rightarrow \theta &\geq -2. \end{aligned}$$

   Hence, from (1) we get that the formula for the pdf of the family is

   $$f(x) = \begin{cases} \theta x + 1 - \frac{\theta}{2} & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

   where $\theta \in [-2, 2]$.

(b) From (2) we get that

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \theta\frac{x^2}{2} + (1 - \frac{\theta}{2})x & x \in [0,1] \\ 1 & x \geq 1. \end{cases} \tag{4}$$

where $\theta \in [-2, 2]$.

(c) Let M_X denote the median of X. We know that $M_X \in [0,1]$ since X takes value only in this interval. Then

$$F(M_X) = \frac{1}{2}.$$
$$\theta M_X^2 + (2 - \theta)M_X - 1 = 0. \tag{5}$$
$$\Rightarrow M_X = \frac{\theta - 2 \pm \sqrt{4 + \theta^2}}{2\theta} \tag{6}$$

Now we need to determine which root is M_X.
Case 1: $\theta \geq 0$.
Notice that since $-4\theta \leq 0$, $4 + \theta^2 \geq (\theta - 2)^2$ and also, $\theta - 2 \leq 0$. Therefore, only the larger root, $\frac{\theta - 2 + \sqrt{4 + \theta^2}}{2\theta} \geq 0$.The other root is negative. Therefore, $M_X = \frac{\theta - 2 + \sqrt{4 + \theta^2}}{2\theta}$.
Case 2: $\theta < 0$.
Notice that since $-4\theta \geq 0$, $4 + \theta^2 \leq (\theta - 2)^2$. This leads to $\theta - 2 + \sqrt{4 + \theta^2} < 0$. Clearly, both the roots are positive since for each of them, denominator and numerator both are negative numbers. Now since $F(x)$ is a strictly increasing function on the interval $[0, 1]$, it will attain the value $\frac{1}{2}$ only at one point in the interval. Hence, only one root of the quadratic in (5) will lie in the interval [0,1]. The other root is clearly greater than 1. Hence, M_X is the smaller root.

$$M_X = \frac{\theta - 2 + \sqrt{4 + \theta^2}}{2\theta}.$$

Hence, in general, for any $\theta \in [-2, 2]$,

$$M_X = \frac{\theta - 2 + \sqrt{4 + \theta^2}}{2\theta}.$$

2. Now consider that continuous random variable X belongs to a two parameter family of all distributions having a linear probability density function (pdf) with slope θ over open intervals $(0, \eta)$.

   (a) Derive the formula for the pdf $f(x)$ of this family, making clear the range of permissible values of θ and η.

   (b) Derive the formula for the cumulative distribution function (cdf) $F(x)$ for this family.

(c) Derive an expression for the median value of X as a function of θ and η.

Ans:

(a) Consider Y to be the random variable described in question 1 with pdf

$$f_Y(y) = \begin{cases} \theta_Y y + 1 - \dfrac{\theta_Y}{2} & y \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Here we denoted the parameter θ_Y instead of θ to avoid confusion with the slope parameter θ of X. Now notice that $Y = \dfrac{X}{\eta}$ or $X = \eta Y$. This follows if we observe that the family of X can be obtained from that of Y by a simple transformation i.e, multiplication by η.
Hence, on $x \in (0, \eta)$,

$$\begin{aligned} f(x) =& \frac{1}{\eta} f_Y(\frac{x}{\eta}) \\ =& \frac{\theta_Y}{\eta^2} x + \frac{2 - \theta_Y}{2\eta} \end{aligned}$$

Therefore, $\theta = \dfrac{\theta_Y}{\eta^2}$. Since the admissible range of θ_Y is $[-2, 2]$, $\theta \in [-\dfrac{2}{\eta^2}, \dfrac{2}{\eta^2}]$.
Hence, on $x \in (0, \eta)$,

$$f(x) = \theta x + \frac{2 - \theta\eta^2}{2\eta}.$$

For $\theta \in [-\dfrac{2}{\eta^2}, \dfrac{2}{\eta^2}]$,

$$f(x) = \begin{cases} \theta x + \dfrac{2 - \theta\eta^2}{2\eta} & x \in (0, \eta) \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

(b) Since $F(0) = 0$, the integration constant will be 0 (like 1.(b)). Therefore,

$$F(x) = \begin{cases} 0 & x \le 0 \\ \theta\dfrac{x^2}{2} + \dfrac{2 - \theta\eta^2}{2\eta} x & x \in (0, \eta) \\ 1 & x \ge \eta. \end{cases} \tag{9}$$

(c) Let M_Y be the median of Y. From 1.(c) we get that,

$$M_Y = \frac{\theta_Y - 2 + \sqrt{4 + \theta_Y^2}}{2\theta_Y}.$$

Since $X = \eta Y$, the median of X, $M_X = \eta M_Y$. Hence,

$$\begin{aligned} M_X =& \eta(\frac{\theta_Y - 2 + \sqrt{4+\theta_Y^2}}{2\theta_Y}). \\ =& \frac{\theta\eta^2 - 2 + \sqrt{4+\theta^2\eta^4}}{2\theta\eta} \end{aligned}$$

replacing θ_Y by $\theta\eta^2$.

3. Suppose random variable X has a negative binomial distribution for some $p \in [0, 1]$ and some integer $r \geq 1$ with

$$Pr[X = x] = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r} & x \in \{r, r+1, \ldots\} \\ 0 & \text{else} \end{cases}$$

Prove that the above formula is a probability mass function.

Ans:

$$\begin{aligned} & \sum_{x=r}^{\infty} Pr[X = x] \\ =& \sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r (1-p)^{x-r} \\ =& \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} p^r (1-p)^k \text{ (replacing } k = x - r) \\ =& \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} p^{r+k} \left(\frac{1-p}{p}\right)^k \\ =& \sum_{k=0}^{\infty} (-1)^k \binom{k+r-1}{r-1} \left(\frac{1}{p}\right)^{-(r+k)} \left(\frac{p-1}{p}\right)^k \\ =& \left(\frac{1}{p} + \frac{p-1}{p}\right)^{-r} \text{ (Ussing the negative binomial series since } \frac{1-p}{p} < \frac{1}{p}). \\ =& 1. \end{aligned}$$

Here we have used the negative binomial expansion for $|x| < a$,

$$(x+a)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k a^{-(n+k)}.$$

Alternative method:

$$\sum_{x=r}^{\infty} Pr[X=x]$$
$$=\sum_{x=r}^{\infty}\binom{x-1}{r-1}p^r(1-p)^{x-r}$$
$$=\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}p^r(1-p)^k \text{ (replacing } k=x-r)$$
$$=(1-p)^{-r}\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}p^{-k}\left(\frac{1}{p(1-p)}\right)^{-(r+k)}$$
$$=(1-p)^{-r}\sum_{k=0}^{\infty}(-1)^k\binom{k+r-1}{r-1}(-\frac{1}{p})^k\left(\frac{1}{p(1-p)}\right)^{-(r+k)}$$
$$=(1-p)^{-r}\left(-\frac{1}{p}+\frac{1}{p(1-p)}\right)^{-r} \text{ (Using the negative binomial series since } \frac{1}{p}<\frac{1}{p(1-p)}).$$
$$=1.$$

Note: That $|x| < a$ is important. If you have not shown that this condition holds in your work, points will be deducted. Moreover, if the terms you choose as x and a do not satisfy $|x| < a$ at all, points will be deducted.

4. Suppose discrete random variable X belongs to the family of all distributions having probability mass function (pmf) of the form

$$p(x) = Pr[X=x] = \begin{cases} \frac{c(\theta)}{\theta^x} & x \in \{0,1,2,\ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

State clearly the range of permissible values of θ and explicitly providing the form of $c(\theta)$. What is the name of this parametric family?

Ans: Clearly $c(\theta) \neq 0$.

$$\sum_{x=0}^{\infty} p(x) = 1.$$
$$\Rightarrow \sum_{x=0}^{\infty}\frac{1}{\theta^x} = c(\theta)^{-1}.$$

$\sum_{x=0}^{\infty}\frac{1}{\theta^x} < \infty$ iff $\theta > 1$. Hence, $\theta \in (1,\infty)$. In this case,

$$\frac{1}{1-\frac{1}{\theta}} = c(\theta)^{-1}.$$
$$c(\theta) = 1-\frac{1}{\theta}.$$

This is the geometric distribution.

5. Suppose continuous random variable X has the exponential distribution $X \sim \mathcal{E}(\lambda)$ with pdf $f(x) = \lambda e^{-\lambda x}\mathbf{1}_{(0,\infty)}(x)$ for $\lambda > 0$.

   (a) What is the cumulative distribution function (cdf) for X?
   (b) What is the median for X?
   (c) For arbitrary positive s and t, find an expression for $Pr(X > s + t|X > s)$.
   (d) Suppose the distribution of X describes the time until failure of a light bulb that is left on continuously. Suppose it is still burning at time s. What is the "median residual lifetime" of the bulb after time s (i.e., the time after s that there is exactly a 50% chance the bulb will still be lit)?

Solution.

(a) The cdf of X for $x > 0$ is

$$\begin{aligned}
F(x) &= Pr(X \le x) \\
&= \int_{-\infty}^{x} f(t)dt \\
&= \int_{0}^{x} \lambda \exp(-\lambda t)dt \\
&= \int_{0}^{\lambda x} \exp(-w)dw \quad \text{(Where } w = \lambda t\text{)} \\
&= [-\exp(-w)]_0^{\lambda x} \\
&= -\exp(-\lambda x) - (-\exp(0)) \\
&= 1 - e^{-\lambda x}, \quad x > 0
\end{aligned}$$

and 0, otherwise.

(b) The median satisfies $F(m) = \frac{1}{2}$, so

$$\begin{aligned}
F(m) &= 1/2 \\
1 - e^{-\lambda m} &= 1/2 \\
\exp(-\lambda m) &= 1/2 \\
-\lambda m &= -\log 2 \\
m &= \lambda^{-1}\log 2.
\end{aligned}$$

(c) This problem demonstrates the unique *memoryless* property of exponential distributions.

$$\begin{aligned}
Pr(X > s+t|X > s) &= \frac{Pr(X > s+t, X > s)}{Pr(X > s)} \\
&= \frac{Pr(X > s+t)}{Pr(X > s)} \\
&= \frac{1 - Pr(X \leq s+t)}{1 - Pr(X \leq s)} \\
&= \frac{1 - F(s+t)}{1 - F(s)} \\
&= \frac{\exp(-\lambda(s+t)}{\exp(-\lambda s)} \\
&= \exp(-\lambda t) \\
&= Pr(X > t).
\end{aligned}$$

Note: a similar exercise shows that the discrete geometric distributions also exhibit this property.

(d) The "median residual lifetime" is the $m > 0$ such that $Pr(X > s+m|X > s) = 1/2$. However, applying the lack of memory property derived in (c),

$$Pr(X > s+m|X > s) = Pr(X > m).$$

Hence m is just the median of X, which was found in (b) to be

$$m = \lambda^{-1} \log 2.$$

6. Suppose continuous random variable X has the standard uniform distribution $X \sim \mathcal{U}(0,1)$ with pdf $f(x) = \mathbf{1}_{(0,1)}(x)$. What are the cumulative distribution function and probability density function for $W = \log(X)$?

Solution. First, we solve the problem as written. A more common transformation in practice would be $Z = -\log(X)$. Working either the problem as written or assuming Z was intended is fine for this homework.
To derive the cdf of W, we first need to note that the cdf of X is just

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{\min(x,1)} 1dt = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}.$$

Then, we calculate

$$\begin{aligned}
F_W(w) &= Pr(W \leq w) \\
&= Pr(\log(X) \leq w) \\
&= Pr(X \leq \exp(w) \\
&= \begin{cases} \exp(w) & w < 0 \\ 1 & w \geq 0 \end{cases}.
\end{aligned}$$

We note that the support of W is now on the interval $(-\infty, 0)$. With this in mind, the pdf of W follows by differentiation:

$$f_W(w) = \frac{dF_W(w)}{dw} = \exp(w)\mathbf{1}_{(-\infty,0)}(w).$$

The alternative formulation of this problem for $Z = -\log(X)$ results in the following cdf:

$$\begin{aligned}
F_Z(z) &= Pr(Z \leq z) \\
&= Pr(-\log(X) \leq z) \\
&= Pr(\log(X) \geq -z) \\
&= Pr(X \geq \exp(-z)) \\
&= Pr(X > \exp(-z)) \quad \text{(by continuity)} \\
&= 1 - Pr(X \leq \exp(-z)) \\
&= \begin{cases} 1 - \exp(-z) & z \geq 0 \\ 0 & z < 0 \end{cases}.
\end{aligned}$$

Hence, Z follows the exponential distribution with $\lambda = 1$ and with pdf $f_Z(z) = \exp(-z)\mathbf{1}_{(0,\infty)}(z)$.

MORE INVOLVED PROBLEMS

7. Suppose independent continuous random variables X_1 and X_2 have the exponential distribution $X \sim \mathcal{E}(\lambda)$ with pdf $f(x) = \lambda e^{-\lambda x}\mathbf{1}_{(0,\infty)}(x)$ for $\lambda > 0$.

   (a) What are the cumulative distribution function and probability density function for $W = \min(X_1, X_2)$?

   (b) What are the cumulative distribution function and probability density function for $W = \max(X_1, X_2)$?

   Solution.

   (a) The cdf calculation should be familiar from Problem 6(a) of Homework #1. Later on, we will study the distributions of arbitrary order statistics (not just the minimum or maximum of a random sample). Here, the distribution of W follows as

   $$\begin{aligned} F(w) = Pr(W \le w) &= 1 - Pr(W > w) \\ &= 1 - Pr(X_1 > w, X_2 > w) \\ &= 1 - Pr(X_1 > w)^2 \\ &= 1 - (1 - Pr(X_1 \le w))^2 \\ &= 1 - (e^{-\lambda w})^2 \\ &= 1 - e^{-2\lambda w} \end{aligned}$$

   for $w > 0$ and 0 for $w \le 0$. By differentiating or noting that $W \sim \mathcal{E}(2\lambda)$, we obtain the pdf

   $$f(w) = 2\lambda \exp(-2\lambda w)\mathbf{1}_{(0,\infty)}(w).$$

   Note: This exercise can be generalized to a sample of $X_1, \ldots, X_n$ that are independent and identically distributed according to $X \sim \mathcal{E}(\lambda)$. What do you expect the distribution of $W = \min(X_1, \ldots, X_n)$ to be, based on this problem?

   (b) For the maximum, we use the independence and identical distribution of the X_i to find:

   $$\begin{aligned} F(w) = Pr(W \le w) &= Pr(X_1 \le w, X_2 \le w) \\ &= Pr(X_1 \le w)^2 \\ &= (1 - \exp(-\lambda w))^2 \end{aligned}$$

   for $w > 0$ and 0 for $w \le 0$. By differentiation of $F(w)$, the pdf of W is

   $$f(w) = 2\lambda(1 - \exp(-\lambda w))\exp(-\lambda w)\mathbf{1}_{(0,\infty)}(w).$$

8. Let X_1 and X_2 be Poisson random variables with $X_i \sim \mathcal{P}(\lambda_i)$ and probability mass functions

   $$p_i(k) = Pr[X_i = x] = \begin{cases} \frac{e^{-\lambda_i}\lambda_i^x}{x!} & x \in \{0, 1, 2, \ldots\} \\ 0 & \text{else} \end{cases}.$$

Further suppose that X_1 and X_2 are independent random variables, so that the events $\{X_1 = x_1\}$ and $\{X_2 = x_2\}$ are independent for all $x_1, x_2 \in \{0, 1, 2, \ldots\}$. *(Note that we will eventually describe methods for deriving these answers using convolutions. However, this problem can be answered by just considering independent events and the results we have discussed for conditional probabilities.)*

(a) What is the joint probability mass function $p_{X_1,X_2}(k_1, k_2) = Pr(X_1 = k_1 \cap X_2 = k_2)$.

(b) Letting random variable $S = X_1 + X_2$, what is the probability mass function $p_S(s)$?

(c) Find the conditional probability mass function $p_{X_1|S}(x|S = s)$. To what parametric family does this family belong?

Solution.

(a) By independence, we have that

$$p_{X_1,X_2}(k_1, k_2) = Pr(X_1 = k_1 \cap X_2 = k_2) = Pr(X_1 = k_1)Pr(X_2 = k_2) = p_1(k_1)p_2(k_2).$$

Hence, the joint distribution of X_1 and X_2 has pmf

$$p_{X_1,X_2}(k_1, k_2) = \begin{cases} \frac{e^{-\lambda_1}\lambda_1^{k_1}}{k_1!}\frac{e^{-\lambda_2}\lambda_2^{k_2}}{k_2!} & k_1, k_2 \in \{0, 1, 2, \ldots\} \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{e^{-(\lambda_1+\lambda_2)}}{k_1!k_2!}\lambda_1^{k_1}\lambda_2^{k_2} & k_1, k_2 \in \{0, 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

(b) We first derive a "useful fact" about $Pr(S = s|X_2 = k)$:

$$\begin{aligned} Pr(S = s|X_2 = k) &= Pr(X_1 + X_2 = s|X_2 = k) \\ &= \frac{Pr(X_1 + X_2 = s, X_2 = k)}{Pr(X_2 = k)} \\ &= \frac{Pr(X_1 + X_2 = (s - k) + k, X_2 = k)}{Pr(X_2 = k)} \\ &= \frac{Pr(X_1 = s - k, X_2 = k)}{Pr(X_2 = k)} \\ &= \frac{Pr(X_1 = s - k)Pr(X_2 = k)}{Pr(X_2 = k)} \quad \text{(by independence)} \\ &= Pr(X_1 = s - k). \end{aligned}$$

Now, we proceed in calculating the mass function of S. For $s =$

$0, 1, 2, \ldots$, we have

$$
\begin{aligned}
p_S(s) &= Pr(S = s) \\
&= \sum_{k=0}^{\infty} Pr(S = s | X_2 = k) Pr(X_2 = k) \quad \text{(by the Law of Total Probability)} \\
&= \sum_{k=0}^{s} Pr(S = s | X_2 = k) Pr(X_2 = k) \quad \text{(since } Pr(k > s) = 0.\text{)} \\
&= \sum_{k=0}^{s} p_1(s-k) p_2(k) \quad \text{(using our useful fact)} \\
&= \sum_{k=0}^{s} \frac{e^{-\lambda_1} \lambda_1^{s-k}}{(s-k)!} \frac{e^{-\lambda_2} \lambda_2^{k}}{k!} \\
&= \sum_{k=0}^{s} \frac{e^{-\lambda_1} \lambda_1^{s-k}}{(s-k)!} \frac{e^{-\lambda_2} \lambda_2^{k}}{k!} \frac{e^{-(\lambda_1+\lambda_2)(\lambda_1+\lambda_2)^s}}{e^{-(\lambda_1+\lambda_2)(\lambda_1+\lambda_2)^s}} \frac{s!}{s!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^s}{s!} \sum_{k=0}^{s} \frac{s!}{(s-k)!k!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{s-k} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^s}{s!} \quad \text{(recognizing the binomial sum)}
\end{aligned}
$$

which is the pmf of a Poisson distribution with mean parameter $\lambda_1 + \lambda_2$.

(c) Before we evaluate the conditional probability for $k \in \{0, 1, 2, \ldots, s-1, s\}$, note for $k \notin \{0, 1, 2, \ldots, s-1, s\}$ that the following probability is zero.

For $k \in \{0, 1, 2, \ldots, s-1, s\}$,

$$
\begin{aligned}
Pr(X_1 = k | S = s) &= \frac{Pr(S = s, X_1 = k)}{Pr(S = s)} \\
&= \frac{Pr(S = s | X_1 = k) Pr(X_1 = k)}{Pr(S = s)} \\
&= \frac{p_2(s-k) p_1(k)}{p_S(s)} \quad \text{(applying the useful fact)} \\
&= \frac{\frac{e^{-\lambda_2} \lambda_2^{s-k}}{(s-k)!} \frac{e^{-\lambda_1} \lambda_1^{k}}{k!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^2}{s!}} \\
&= \frac{s!}{(s-k)!k!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{s-k}.
\end{aligned}
$$

Equivalently, we could write

$$
Pr(X_1 = k | S = s) = \frac{s!}{(s-k)!k!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{s-k} \mathbf{1}_{\{0,1,2,\ldots,s\}}(k),
$$

which emphasizes for arbitrary $k \in \mathbb{R}$ whether k is in the support of the conditional distribution or not.

This is the pmf of a Binomial distribution with probability of success $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ in a sample of size s. We can also write

$$X_1|S = s \sim \mathcal{B}\left(s, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right).$$

9. Consider an experiment consisting of 2 independent rolls of a weighted die. Let X_k be the random variable measuring the number showing on top of the die on the kth roll. Hence, the sample space for X_k is $\Omega_X = \{1, 2, 3, 4, 5, 6\}$, with $P_X(X_k = \omega) = p_\omega$ for all $\omega \in \Omega_X$ and for $k = 1, 2$. Define $S = X_1 + X_2$.

   (a) What is the support of S?

   (b) Can you find values for the parameters $p_1, \ldots, p_6$ such that all possible outcomes for S are equally likely? If so, do so. If not, prove that it cannot be done.

Solution.

(a) The support of S is the set of values s such that $P_S(S = s) > 0$. Since the probabilities for each outcome in Ω_X must be positive, the support of S is given by $\Omega_S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. This is the set of all possible outcomes of S.

(b) In order for the outcomes of S to be equally likely, we require that $P_S(S = s) = P(s) = 1/11$ for all $s \in \Omega_S$. Thus, the following *must* hold:

$$P(2) = p_1^2 = 1/11$$
$$P(12) = p_6^2 = 1/11$$

Then we have that $p_1 = p_6 = 1/\sqrt{11}$. However, this would imply that

$$P(7) = 2(p_1p_6 + p_2p_5 + p_3p_4) = 2(1/11 + p_2p_5 + p_3p_4) > 1/11$$

since $p_\omega > 0$ for each $\omega \in \Omega_X$. This suffices to show that no weighted die can be constructed such that the outcomes of S are all equally likely. Note that additional correct solutions may be constructed based on outcomes other than $\omega = 1$ or 6.