Stat 512
Emerson, Fall 15
Homework #5
November 2 2015

Written solutions to the homework problems are due on Monday, November 9, 2015 at the beginning of class.

The homework problems are divided into "regular" and "more involved" problems. In order to facilitate multiple graders, you should hand in these categories of problems separately. That is, hand in one paper that contains only the "regular" problems, and another paper that contains only the "more involved" problems.
As noted on the syllabus and discussed during the first class, copying of homework solutions is not allowed and, when detected, will be investigated as an infraction of the academy integrity policy of the University of Washington. While it is permissible to discuss problems with other students, TAs, or the instructor in order to learn how to solve a problem, your written solutions must be prepared without directly referencing any notes or solutions derived from other students or sources found on the internet.

REGULAR PROBLEMS

1. Let X be a random variable with a beta distribution having parameters $\alpha > 0$ and $\beta > 0$ with probability distribution function

$$f_X(x \mid \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{(0,1)}(x),$$

where $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ is the beta function. Explicitly derive $E[X]$ and $Var(X)$.
Recall the useful relationships between beta and gamma functions ($\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}\,du$) for real $\alpha > 0$ and $\beta > 0$ and integer $n > 0$:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad \Gamma(n) = (n-1)!$$

2. Let $L \sim \mathcal{U}(0, \alpha)$ and $W \sim \mathcal{U}(0, \beta)$ be independent random variables measuring the length and width, respectively, of a rectangle. Let $A = LW$ be the random variable measuring the area of a rectangle.

   (a) Find the probability distribution for A.

(b) Find $E[A]$ and $Var(A)$.

(c) Now suppose that $L \sim (\mu, \sigma^2)$ and $W \sim (\nu, \tau^2)$ are two independent random variables with mean and variance as specified, but whose distributions are otherwise unspecified. Find $E[A]$ and $Var(A)$ in this more general setting.

(d) Again consider that more general setting, but now suppose that $\text{corr}(L, W) = \rho$. Find $E[A]$. Can you find $Var(A)$? If so, do so. If not, explain what additional information you would need.

3. Let X_1, X_2 be independent, identically distributed random variables, each having a uniform distribution: $X_i \sim \mathcal{U}(0, \theta)$.

   (a) For $W = (X_1 + X_2)/2$, find the probability distribution function $f_W(w \,|\, \theta)$ and $E[W]$ and $Var(W)$.

   (b) For what value of a would "estimator" $\hat{\theta}_W = aW$ be unbiased (i.e., have $E[\hat{\theta}_W] = \theta$)? What is $Var(\hat{\theta}_W)$ for that value of a?

   (c) For $Y = \max(X_1, X_2)$, find the probability distribution function $f_Y(y \,|\, \theta)$ and $E[Y]$ and $Var(Y)$.

   (d) For what value of b would "estimator" $\hat{\theta}_Y = bW$ be unbiased (i.e., have $E[\hat{\theta}_Y] = \theta$? What is $Var(\hat{\theta}_Y)$ for that value of b?

   (e) Which of the unbiased estimators $\hat{\theta}_W$ and $\hat{\theta}_Y$ are more precise (i.e., have lower variance)?

4. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables having the uniform distribution $X_i \sim \mathcal{U}(0, \theta)$.

   (a) Derive the moment generating function $M_X(t)$ for X.

   (b) Derive the moment generating function $M_Y(t)$ for $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$

5. Let $X_1, X_2, \ldots, X_n$ be independent random variables having Poisson distributions $X_i \sim \mathcal{P}(\lambda_i)$.

   (a) Derive the moment generating function $M_{X_i}(t)$ for X_i.

   (b) Derive the moment generating function $M_Y(t)$ for $Y = \sum_{i=1}^{n} X_i$ and thus identify the probability distribution for Y.

MORE INVOLVED PROBLEMS

6. Let X_{ij} be the jth measurement made on the ith subject in some experiment, with $j = 1, \ldots, r$ and $i = 1, \ldots, n$. Suppose the X_{ij} are identically distributed with $E[X_{ij}] = \mu$ and $Var(X_{ij}) = \sigma^2$. Further suppose that every individual is independent of one

another, but that measurements made on the same individual are correlated with correlation ρ. That is,

$$corr(X_{ij}, X_{k\ell}) = \mathbf{1}_{[i=k]} \left(\mathbf{1}_{[j=\ell]} + \rho \mathbf{1}_{[j \neq \ell]} \right).$$

Let $\overline{X}_{i\cdot} = \sum_{j=1}^{r} X_{ij}/r$, and let $\overline{X}_{\cdot\cdot} = \sum_{i=1}^{n} \overline{X}_{i\cdot}/n$.

(a) Derive $E[\overline{X}_{i\cdot}]$.

(b) Derive $Var(\overline{X}_{i\cdot})$.

(c) Derive $E[\overline{X}_{\cdot\cdot}]$.

(d) Derive $Var(\overline{X}_{\cdot\cdot})$.