Stat 512
Homework key 5

November 9, 2015

REGULAR PROBLEMS

1. Let X be a random variable with a beta distribution having parameters $\alpha > 0$ and $\beta > 0$ with probability distribution function
$$f_X(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{(0,1)}(x),$$
where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ is the beta function. Explicitly derive $E[X]$ and $Var(X)$.
Recall the useful relationships between beta and gamma functions ($\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}du$) for real $\alpha > 0$ and $\beta > 0$ and integer $n > 0$:
$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad \Gamma(n) = (n-1)!$$

Ans:

For integer $m > 0$,
$$\begin{aligned}
E(X^m) &= \int_0^1 x^m f_X(x|\alpha,\beta)dx \\
&= \int_0^1 \frac{1}{B(\alpha,\beta)} x^{(\alpha+m)-1}(1-x)^{\beta-1}dx \\
&= \frac{B(\alpha+m,\beta)}{B(\alpha,\beta)} \int_0^1 \frac{1}{B(\alpha+m,\beta)} x^{(\alpha+m)-1}(1-x)^{\beta-1}dx \\
&= \frac{B(\alpha+m,\beta)}{B(\alpha,\beta)} \\
&= \frac{\Gamma(\alpha+m)\Gamma(\beta)}{\Gamma(\alpha+m+\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\
&= \frac{(\alpha+m-1)(\alpha+m-2)\cdots\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+m+\beta-1)(\alpha+m-2)\cdots(\alpha+\beta)\Gamma(\alpha+\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\
&= \frac{(\alpha+m-1)(\alpha+m-2)\cdots\alpha}{(\alpha+m+\beta-1)(\alpha+m-2)\cdots(\alpha+\beta)}.
\end{aligned}$$

Hence we have for $m = 1, 2$:
$$E(X) = \frac{\alpha}{\alpha+\beta},$$
$$E(X^2) = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}.$$

We can now use a classic formula for the variance,
$$\begin{aligned}
Var(X) &= E(X^2) - (E(X))^2 \\
&= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{\alpha}{\alpha+\beta}\left(\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right) \\
&= \frac{\alpha}{\alpha+\beta}\left(\frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)}\right) \\
&= \frac{\alpha}{\alpha+\beta}\left(\frac{\beta}{(\alpha+\beta+1)(\alpha+\beta)}\right) \\
&= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.
\end{aligned}$$

2. Let $L \sim \mathcal{U}(0, \alpha)$ and $W \sim \mathcal{U}(0, \beta)$ be independent random variables measuring the length and width, respectively, of a rectangle. Let $A = LW$ be the random variable measuring the area of a rectangle.

   (a) Find the probability distribution for A.

   (b) find $E(A)$ and $Var(A)$.

   (c) Now suppose that $L \sim (\mu, \sigma^2)$ and $W \sim (\nu, \tau^2)$ are two independent random variables with mean and variance as specified, but whose distributions are otherwise unspecified. Find $E[A]$ and $Var(A)$ in this more general setting.

   (d) Again consider that more general setting, but now suppose that$Corr(L, W) = \rho$. Find $E[A]$. Can you find $Var(A)$? If so, do so. If not, explain what additional information you would need.

Ans:

(a) Let the density of A be f_A. We denote the density of L and W by f_L and f_W respectively. Then for $x \leq \alpha\beta$,

$$\begin{aligned} f_A(x) &= \int_{-\infty}^{\infty} \frac{1}{l} f_{L,W}(l, \frac{x}{l}) dl \\ &= \int_{-\infty}^{\infty} \frac{1}{l} f_L(l) f_W(\frac{x}{l}) dl. \end{aligned}$$

since L and W are independent. Now $f_L(l)$ and $f_W(\frac{x}{l})$ both are positive only if $0 < l \leq \alpha$ and $0 < \frac{x}{l} \leq \beta$. Therefore,

$$\begin{aligned} f_A(x) &= \int_{x/\beta}^{\alpha} \frac{1}{l\alpha\beta} dl \\ &= \frac{1}{\alpha\beta}\left[\log\alpha - \log\frac{x}{\beta}\right] \\ &= \frac{1}{\alpha\beta}\log\frac{\alpha\beta}{x} \end{aligned}$$

Hence,

$$f_A(x) = \frac{1}{\alpha\beta}\log\frac{\alpha\beta}{x} 1[0 < x \leq \alpha\beta].$$

(b)

$$E[A] = E[L][W] = \frac{\alpha\beta}{4}$$

since L, W are independent.

$Var(A) = EA^2 - (EA)^2 = EL^2W^2 - \frac{\alpha^2\beta^2}{16}$. Now, $EL^2 = \int_0^\alpha \frac{l^2}{\alpha} dl = \frac{\alpha^2}{3}$.

Therefore,

$$\begin{aligned} &Var(A) \\ &= EL^2W^2 - \frac{\alpha^2\beta^2}{16} \\ &= \frac{\alpha^2\beta^2}{9} - \frac{\alpha^2\beta^2}{16} \\ &= \frac{7\alpha^2\beta^2}{144}. \end{aligned}$$

(c)

$$EA = ELW = EL.EW = \mu\nu$$

since L and W are independent.

$$\begin{aligned} &Var(A) \\ &= EL^2W^2 - (ELW)^2 \\ &= EL^2EW^2 - (ELW)^2 \\ &= (Var(L) + (EL)^2)(Var(W) + (EW))^2 - (\mu\nu)^2 \\ &= (\sigma^2 + \mu^2)(\tau^2 + \nu^2) - (\mu\nu)^2 \\ &= \sigma^2\tau^2 + \sigma^2\nu^2 + \mu^2\tau^2 \end{aligned}$$

(d)

$$\rho = \frac{ELW - EL.EW}{\sqrt{Var(L)Var(W)}} \Rightarrow EA = ELW = \rho\sigma\tau + \mu\nu.$$

But to get $Var(A)$ we need to know EL^2W^2. Therefore, we can not find $Var(A)$ using this information.

3. Let X_1, X_2 be independent, identically distributed random variables, each having a uniform distribution: $X_i \sim \mathcal{U}(0, \theta)$.
   (a) For $W = (X_1 + X_2)/2$, find the probability distribution function $f_W(w|\theta)$ and $E[W]$ and $Var(W)$.
   (b) For what value of a would "estimator" $\hat{\theta}_W = aW$ be unbiased (i.e., have $E[\hat{\theta}_W] = \theta$)? What is $Var(\hat{\theta}_W)$ for that value of a?
   (c) For $Y = \max(X_1, X_2)$, find the probability distribution function $f_Y(y|\theta)$ and $E[Y]$ and $Var(Y)$.
   (d) For what value of b would "estimator" $\hat{\theta}_Y = bY$ be unbiased? What is $Var(\hat{\theta}_Y)$ for that value of b?
   (e) Which of the unbiased estimators $\hat{\theta}_W$ and $\hat{\theta}_Y$ are more precise (i.e., have lower variance)?

Ans:

(a) Let $Z = X_1 + X_2$. As seen in Homework 4, Problem 3(c),

$$f_Z(z) = \frac{z}{\theta^2}\mathbf{1}_{(0,\theta]}(z) + \frac{2\theta - z}{\theta^2}\mathbf{1}_{(\theta,2\theta]}(z).$$

Then $W = Z/2$ and $Z = 2W$, so that by our result about monotone transformations of random variables (or location-scale families):

$$f_W(w) = 2f_Z(2w) = \frac{4}{\theta^2}\left(w\mathbf{1}[0 < w \leq \theta/2] + (\theta - w)\mathbf{1}[\theta/2 < w \leq \theta]\right).$$

The expectation and variance follow easily:

$$E(W) = \frac{1}{2}E(X_1 + X_2) = E(X_1) = \frac{\theta}{2},$$

$$Var(W) = \frac{1}{4}(VarX_1 + Var_2) = \frac{1}{2}VarX_1 = \frac{\theta^2}{24}.$$

(b) We have that

$$E(aW) = aE(W) = \frac{a\theta}{2},$$

so $a = 2$ results in an unbiased estimator, $\hat{\theta}_W$, having variance

$$Var(\hat{\theta}_W) = 4Var(W) = \frac{\theta^2}{6}.$$

(c) Familiar calculations about the cdf of the sample maximum lead us to

$$F_Y(y) = F_X(y)^2\mathbf{1}[0 \leq y < \theta] + \mathbf{1}[\theta \leq y] = \frac{y^2}{\theta^2}\mathbf{1}[0 \leq y < \theta] + \mathbf{1}[\theta \leq y].$$

Thus the pdf of Y is just

$$f_Y(y) = \frac{2y}{\theta^2}\mathbf{1}[0 \leq y < \theta].$$

For integer $m > 0$,

$$E(Y^m) = \int_0^\theta 2\frac{y^{m+1}}{\theta^2}dy = \frac{2}{m+2}\theta^m.$$

Hence, $E(Y) = \frac{2}{3}\theta$ and $E(Y^2) = \frac{1}{2}\theta^2$. This yields

$$Var(Y) = E(Y^2) - (EY)^2 = \frac{1}{2}\theta^2 - \frac{4}{9}\theta^2 = \frac{1}{18}\theta^2.$$

(d) For $b = 3/2$, by linearity of expectation, we see that the estimator $\hat{\theta}_Y$ is unbiased:

$$E(\hat{\theta}_Y) = bE(Y) = \frac{3}{2}\frac{2}{3}\theta = \theta.$$

Further since this estimator is a multiple of Y, the variance is just

$$Var(\hat{\theta}_Y) = \frac{9}{4}\frac{\theta^2}{18} = \frac{\theta^2}{8}.$$

(e) We see that for the unbiased estimators in question, $\hat{\theta}_Y$ has lower variance (is more precise) than $\hat{\theta}_W$:

$$Var(\hat{\theta}_Y) = \frac{\theta^2}{8} < \frac{\theta^2}{6} = Var(\hat{\theta}_W).$$

4. Let $X_1, \ldots, X_n$ be independent, identically distributed random variables having the uniform distribution $X_i \sim \mathcal{U}(0, \theta)$.

(a) Derive the moment generating function$M_X(t)$ for X.

(b) Derive the moment generating function $M_Y(t)$ for $Y = \frac{1}{n}\sum_{i=1}^n X_i$

Ans:

(a) Let $t = 0$. Then $M_X(t) = Ee^{tX} = E1 = 1$. Now suppose $t \neq 0$. Then

$$\begin{aligned} M_X(t) = Ee^{tX} &= \int_0^\theta \frac{e^{tx}}{\theta} dx \\ &= \frac{e^{t\theta} - 1}{t\theta}. \end{aligned}$$

Hence, $M_X(t) = \begin{cases} \dfrac{e^{t\theta}-1}{t\theta} & \text{if } t \neq 0 \\ 1 & o.w. \end{cases}$ We see that $M_X(t)$ has a discontinuity at 1 (removable discontinuity as the limit exists at 0.)

(b) Since $X_1, \ldots, X_n$ are i.i.d,

$$\begin{aligned} M_Y(t) &= Ee^{\sum_{i=1}^n X_i/n} \\ &= \prod_{i=1}^n Ee^{tX_i/n} \\ &= \prod_{i=1}^n M_{X_i}\left(\frac{t}{n}\right) \\ &= \prod_{i=1}^n \frac{e^{t\theta/n} - 1}{\frac{t\theta}{n}} \\ &= \left(\frac{n(e^{t\theta/n} - 1)}{t\theta}\right)^n \end{aligned}$$

5. Let $X_1, X_2, \ldots, X_n$ be independent random variables having Poisson distributions $X_i \sim \mathcal{P}(\lambda_i)$.

(a) Derive the moment generating function $M_{X_i}(t)$ for X_i.

(b) Derive the moment generating function $M_Y(t)$ for $Y = \sum_{i=1}^n X_i$ and thus identify the probability distribution of Y.

Ans:

(a) For a given $X \sim \mathcal{P}(\lambda)$ (where we suppress the subscript i),

$$\begin{aligned} M_X(t) &= E(\exp(tX)) \\ &= \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\lambda} \lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} \frac{e^{-\lambda} (\lambda e^t)^x}{x!} \\ &= \exp(\lambda(e^t - 1)) \sum_{x=0}^{\infty} \frac{e^{-\lambda e^t} (\lambda e^t)^x}{x!} \\ &= \exp(\lambda(e^t - 1)). \end{aligned}$$

Hence, $M_{X_i}(t) = \exp(\lambda_i(e^t - 1))$.

(b) We make use of the independence of the X_i to factor the MGF of Y as follows:

$$\begin{aligned} M_Y(t) &= E(\exp(tY)) \\ &= E[\exp(t \sum_{i=1}^{n} X_i)] \\ &= E\left[\prod_{i=1}^{n} \exp(tX_i)\right] \\ &= \prod_{i=1}^{n} E[\exp(tX_i)] \\ &= \prod_{i=1}^{n} \exp(\lambda_i(e^t - 1)) \\ &= \exp((\sum_{i=1}^{n} \lambda_i)(e^t - 1)). \end{aligned}$$

We recognize this to be the MGF of a Poisson distribution. Hence, $Y \sim \mathcal{P}(\sum_{i=1}^{n} \lambda_i)$.

MORE INVOLVED PROBLEM

6. Let X_{ij} be the jth measurement made on the ith subject in some experiment, with $j = 1, \ldots, r$ and $i = 1, \ldots, n$ Suppose the X_{ij} are identically distributed with $E[X_{ij}] = \mu$ and $Var(X_{ij}) = \sigma^2$. Further suppose that every individual is independent of one another, but that measurements made on the same individual are correlated with correlation ρ. That is,

$$corr(X_{ij}, X_{kl}) = 1_{[i=k]}\left(1_{[j=l]} + \rho 1_{[j\neq l]}\right).$$

Let $\bar{X}_{i.} = \dfrac{\sum_{j=1}^{r} X_{ij}}{r}$, and let $\bar{X}_{..} = \dfrac{\sum_{i=1}^{n} \bar{X}_{i.}}{n}$.

(a) Derive $E[\bar{X}_{i.}]$.

(b) Derive $Var(\bar{X}_{i.})$.

(c) Derive $E[\bar{X}_{..}]$.

(d) Derive $Var(\bar{X}_{..})$.

Ans:

(a)

$$\begin{aligned} E[\bar{X}_{i.}] &= E\frac{\sum_{j=1}^{r} X_{ij}}{r} = \frac{\sum_{j=1}^{r} EX_{ij}}{r} \\ &= \mu. \end{aligned}$$

(b)

$$\begin{aligned} Var(\bar{X}_{i.}) &= Var\left(\frac{\sum_{j=1}^{r} X_{ij}}{r}\right) \\ &= \sum_{j=1}^{r} Var\left(\frac{X_{ij}}{r}\right) + 2\sum_{1\leq j<j'\leq r} cov\left(\frac{X_{ij}}{r}, \frac{X_{ij'}}{r}\right) \\ &= \frac{\sigma^2}{r} + \frac{r(r-1)\rho\sigma^2}{r^2} \end{aligned}$$

Notice that, $cov(X_{ij}, X_{ij'}) = corr(X_{ij}, X_{ij'})\sqrt{Var(X_{ij})Var(, X_{ij'})} = \rho\sigma^2$. Hence,

$$Var(\bar{X}_{i.}) = \frac{\sigma^2}{r}(1 + (r-1)\rho).$$

(c)

$$E\bar{X}_{..} = \frac{\sum_{i=1}^{n} E\bar{X}_{i.}}{n} = \mu.$$

(d) Since X_{ij} is independent of $X_{kj'}$, for $i \neq k$, $\bar{X}_{i.}$-s are iid. Therefore,

$$Var(\bar{X}_{..}) = \frac{Var(\bar{X}_{i.})}{n} = \frac{\sigma^2}{rn}(1 + (r-1)\rho).$$