Stat 512
Emerson, Fall 15
Homework #7
November 29 2015

Written solutions to the homework problems are due on Friday, December 4, 2015 at the beginning of class.

The homework problems are divided into "regular" and "more involved" problems. In order to facilitate multiple graders, you should hand in these categories of problems separately. That is, hand in one paper that contains only the "regular" problems, and another paper that contains only the "more involved" problems.
As noted on the syllabus and discussed during the first class, copying of homework solutions is not allowed and, when detected, will be investigated as an infraction of the academy integrity policy of the University of Washington. While it is permissible to discuss problems with other students, TAs, or the instructor in order to learn how to solve a problem, your written solutions must be prepared without directly referencing any notes or solutions derived from other students or sources found on the internet.

REGULAR PROBLEMS

1. Let $\hat{\theta}_1, \ldots, \hat{\theta}_p$ be independent estimators of θ with sampling distributions $\hat{\theta}_k \sim (\theta, V_k)$. Consider linear combination of these estimators

$$\hat{\theta} = \sum_{k=1}^{p} w_k \hat{\theta}_k.$$

   (a) Find the mean and variance of $\hat{\theta}$.

   (b) Specify necessary and sufficient conditions on $\vec{w} = (w_1, \ldots, w_p)$ such that $\hat{\theta}$ is unbiased for θ.

   (c) Find the optimal values of $\vec{w}$ such that $\hat{\theta}$ is the "best linear unbiased estimator" in that it is unbiased and has lower variance than any other unbiased estimator.

2. Let $Y_1, \ldots, Y_n$ be independent random variables having distribution $Y_i \sim (\mu_i, \sigma^2)$ with $\mu_i = \beta_0 + \beta_1 x_i$ for known constants $x_1, \ldots, x_n$ satisfying $\exists i, j \ s.t. \ x_i \neq x_j$. (This is of course the setting of simple linear regression.) One intuitive class of estimators that has been extensively studied is the "least squares estimator" (LSE) such that $\hat{\vec{\mu}}$ minimizes the "sum of squares"

$$\hat{\vec{\mu}} = \text{argmin} \sum_{i=1}^{n} (Y_i - \hat{\mu}_i)^2,$$

which in turn leads to LSE $\hat{\vec{\beta}} = (\hat{\beta}_0, \hat{\beta}_1)$ minimizing

$$\sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 x_i)^2.$$

(a) Find a closed form expression for slope LSE $\hat{\beta}_1$.

(b) Find the mean and variance of the sampling distribution for $\hat{\beta}_1$.

(c) Find a closed form expression for intercept LSE $\hat{\beta}_0$.

(d) Find the mean and variance of the sampling distribution for $\hat{\beta}_0$.

(e) Find the covariance of $\hat{\beta}_0$ and $\hat{\beta}_1$.

3. Now let $Y_1, \ldots, Y_n$ be independent random variables having normal distribution $Y_i \sim \mathcal{N}(\mu_i, \sigma^2)$ with $\mu_i = \beta_0 + \beta_1 x_i$ for known constants $x_1, \ldots, x_n$ satisfying $\exists i, j \; s.t. \; x_i \neq x_j$. (This is of course the setting of simple linear regression.) Another class of estimators that has been extensively studied is the "maximum likelihood estimator" (MLE) of regression parameter $\vec{\beta}$ found by maximizing the likelihood function

$$L(\vec{\beta} \,|\, \vec{Y}) = f_{\vec{Y}}(\vec{Y} \,|\, \vec{\beta}),$$

(that is, the likelihood is just the joint density of the data evaluated at the observed data and regarded as a function of the unknown parameter $\vec{\beta}$). with MLE $\hat{\vec{\beta}}$ defined by

$$\hat{\vec{\beta}} = \arg\max L(\vec{\beta} \,|\, \vec{Y}).$$

Note that when the support of $\vec{Y}$ is independent of $\vec{\beta}$, we most often find the MLE by considering the maximization of the log likelihood

$$\mathcal{L}(\vec{\beta} \,|\, \vec{Y}) = \log\left(f_{\vec{Y}}(\vec{Y} \,|\, \vec{\beta})\right).$$

(a) What is the likelihood function $L(\vec{\beta} \,|\, \vec{Y})$ for this problem?

(b) What is the log likelihood function $\mathcal{L}(\vec{\beta} \,|\, \vec{Y})$ for this problem?

(c) Find a closed form expression for slope MLE $\hat{\beta}_1$.

(d) Find the sampling distribution for $\hat{\beta}_1$.

(e) Find a closed form expression for intercept MLE $\hat{\beta}_0$.

(f) Find the sampling distribution for $\hat{\beta}_0$.

(g) Find the covariance of $\hat{\beta}_0$ and $\hat{\beta}_1$.

(h) Find the joint sampling distribution for $\hat{\vec{\beta}}$.

4. Let $Y_1, \ldots, Y_n$ be independent, identically distributed random variables having log normal distribution $Y_i \sim \mathcal{LN}(\mu, \sigma^2)$ with σ^2 known.
   (a) Find a method of moments estimator (MME) for μ and provide its asymptotic sampling distribution.
   (b) Derive a maximum likelihood estimate for μ and provide its asymptotic sampling distribution.
   (c) Derive which of the two above estimators is more precise.

5. Let $Y_1, \ldots, Y_n$ be independent, identically distributed random variables having normal distribution $Y_i \sim \mathcal{N}(\mu, \sigma^2)$ with σ^2 known. Let $\theta = Pr(Y_i \leq 0)$ be a target of inference.
   (a) Derive a maximum likelihood estimate for μ and provide its asymptotic sampling distribution.
   (b) Derive a maximum likelihood estimate for θ and provide its asymptotic sampling distribution.
   (c) Let $W_i = \mathbf{1}_{(-\infty,0]}(Y_i)$ be an indicator that Y_i is less than or equal to 0, and derive an estimator of θ based on the W_i's and provide its asymptotic sampling distribution.
   (d) Derive which of the two above estimators is more precise.

MORE INVOLVED PROBLEMS

6. Let $Y_1, \ldots, Y_n$ be independent random variables having distribution $Y_i \sim (\mu_i, \sigma^2)$ with $\mu_i = \beta_0 + \beta_1 x_i$ for known constants $x_1, \ldots, x_n$ satisfying $x_i \neq x_j$ for $i \neq j$. For $i \neq j$ define
$$\hat{\theta}_{ij} = \frac{(Y_i - Y_j)}{(x_i - x_j)}$$
.
   (a) Find the mean and variance of the sampling distribution of $\hat{\theta}_{ij}$.
   (b) Using the results of problem 1, find the values of a_{ij} such that
$$\hat{\theta} = \sum_{i=1}^{n} \sum_{j \neq i} a_{ij} \hat{\theta}_{ij}$$
   is the best linear unbiased estimator of $\theta = \beta_1$.
   (c) Explicitly show that $\hat{\theta}$ is exactly the least squares estimate for β_1 as derived in problem 2 above.