Stat 512
Emerson, Fall 15
Homework #8
December 4, 2015

Written solutions to the homework problems are due on Friday, December 11, 2015 at the beginning of class.

The homework problems are divided into "regular" and "more involved" problems. In order to facilitate multiple graders, you should hand in these categories of problems separately. That is, hand in one paper that contains only the "regular" problems, and another paper that contains only the "more involved" problems.
As noted on the syllabus and discussed during the first class, copying of homework solutions is not allowed and, when detected, will be investigated as an infraction of the academy integrity policy of the University of Washington. While it is permissible to discuss problems with other students, TAs, or the instructor in order to learn how to solve a problem, your written solutions must be prepared without directly referencing any notes or solutions derived from other students or sources found on the internet.

REGULAR PROBLEMS

1. Let $Y_1, \ldots, Y_n$ be independent identically distributed random variables according to a Rayleigh distribution with parameter $\sigma^2 > 0$ and probability density function

$$f_Y(y \mid \sigma^2) = \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}} \mathbf{1}_{(0,\infty)}(x).$$

   (a) Derive the cumulative distribution function F_Y and the cdf F_W for $W = Y^2$.
   (b) Derive $E[Y]$.
   (c) Derive $Var(Y)$.
   (d) Find a method of moments estimator (MME) for σ^2 using the first moment. Find its asymptotic distribution.
   (e) Find a method of moments estimator (MME) for σ^2 using the second moment. Find its asymptotic distribution.
   (f) Find a maximum likelihood estimator (MLE) for σ^2. Find its asymptotic distribution.
   (g) Which of the above estimators is to be preferred? Why?

2. Let $Y_1, \ldots, Y_n$ be independent identically distributed random variables according to a one parameter exponential family with parameter θ. For each of the following distributions, find an MME $\tilde{\theta}$ of θ and its asymptotic distribution, as well as the MLE $\hat{\theta}$ of θ and its asymptotic distribution.

(a) Bernoulli distribution: $Y_i \sim \mathcal{B}(1, \theta)$.

(b) Poisson distribution: $Y_i \sim \mathcal{P}(\theta)$.

(c) Exponential distribution: $Y_i \sim \mathcal{E}(\theta)$ with $F_Y(y) = (1 - e^{-y/\theta})\mathbf{1}_{(0,\infty)}(y)$.

3. For each of the estimators found in problems 1 and 2, there was a mean-variance relationship: The variance of the asymptotic distribution of our estimator involved the unknown parameter:
$$\sqrt{n}(T_n(\vec{Y}) - \theta) \to_d \mathcal{N}(0, V(\theta)\,).$$
This sometimes creates a problem when trying to find confidence intervals for θ based on the asymptotic distribution. One approach that is sometimes used to use the delta method to find a "variance stabilizing transformation" $g_{F,\hat{\theta}}(t)$ for each distribution function F and estimator $\hat{\theta}$ such that
$$\sqrt{n}(g(T_n) - g(\theta)) \to_d \mathcal{N}(0, V^*)\,),$$
where V^* is independent of θ.

   (a) Find a variance stabilizing transformation for the estimator found in problem 1d.

   (b) Find a variance stabilizing transformation for the estimator found in problem 1e.

   (c) Find a variance stabilizing transformation for the estimator found in problem 1f.

   (d) Find a variance stabilizing transformation for the estimator found in problem 2a.

   (e) Find a variance stabilizing transformation for the estimator found in problem 2b.

   (f) Find a variance stabilizing transformation for the estimator found in problem 2c.

   Note: The use of these variance stabilizing transformations will at times help us in the following manner:

   - *We use the resulting asymptotic normal distribution to construct an approximate distribution for $g(T_n)$:*
$$g(T_n) \dot{\sim} \mathcal{N}\left(g(\theta),\ \frac{V^*}{n}\right)$$
   - *Based on observation $T_n = t$, we construct a confidence interval for $g(\theta)$ using the usual approach for a normal distribution:*
$$100(1-\alpha)\%\ CI\ for\ g(\theta) = (L,\, U) = \left(g(t) - z_{1-\alpha/2}\sqrt{\frac{V^*}{n}},\ g(t) + z_{1-\alpha/2}\sqrt{\frac{V^*}{n}}\right).$$
   - *Then for invertible g, we compute*
$$100(1-\alpha)\%\ CI\ for\ \theta = (g^{-1}(L),\, g^{-1}(U)\,).$$

4. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables having mean $\mu > 0$ and variance $\sigma^2 > 0$, and let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables having mean $\nu > 0$ and variance $\tau^2 > 0$. Further suppose that X_i and Y_j are independent if $i \neq j$, but that the correlation between X_i and Y_j is ρ if $i = j$. For notational convenience, denote $\vec{X}_n = (X_1, \ldots, X_n)$ and $\vec{Y}_n = (Y_1, \ldots, Y_n)$.

(a) Find a method of moments estimator $\hat{\theta}_n = \hat{\theta}_n(\vec{X}_n, \vec{Y}_n)$ for $\theta = \mu - \nu$ and derive its asymptotic distribution. Is your estimator unbiased? Consistent?

(b) Find a method of moments estimator $\hat{\psi}_n = \hat{\psi}_n(\vec{X}_n, \vec{Y}_n)$ for $\psi = \mu/\nu$ and derive its asymptotic distribution. Is your estimator unbiased? Consistent?

(c) Define $W_i = X_i/Y_i$. Let $\gamma = E[W_i]$. Show that a MME estimator $\hat{\gamma}_n = \hat{\gamma}_n(\vec{X}_n, \vec{Y}_n)$ of γ is in general a biased, inconsistent estimator for ψ.

MORE INVOLVED PROBLEMS

5. (This problem draws heavily on problems 3 and 4 on Homework 3) Let $\vec{X} = (X_1, \ldots, X_n)$ be a random vector in which the X_i are independently distributed with a one exponential family distribution having canonical parameter η and density (probability mass function) of the form

$$f_X(x \mid \eta) = h(x) \exp\left[T(x)\eta - A(\eta)\right].$$

Assume that we may twice interchange integration of f_X with respect to x and differentiation of f_X with respect to η and that $A(\eta)$ is twice differentiable with invertible derivatives.

(a) Find the asymptotic distribution of the maximum likelihood estimate $\hat{\eta}$ of η.

(b) Find the asymptotic distribution of the MLE $\hat{\theta}$ of $\theta = E[X]$ if $\eta = g(\theta)$.