Stat 512
Midterm Examination
November 10, 2015

Closed book, closed notes.

You must stop work precisely when you are told time is up. No paper will be accepted from a student still writing after the class has been told the exam is over.

In addition to the theorems covered in class, you may use the following facts without proof:

- If $X \sim \mathcal{E}(\lambda)$ (an exponential random variable with density $f_X(x) = \lambda e^{-\lambda x}\mathbf{1}_{[0<x<\infty]}$ for some $\lambda > 0$), then $EX = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$ is a normal random variable, then $EX = \mu$, $Var(X) = \sigma^2$, and the third and fourth central moments are 0, and $3\sigma^4$, respectively.

1. (35 points) Let X be a random variable having density function for some constant $a > 0$

$$f_X(x) = a(x-4)^2\mathbf{1}_{[1<x<4]}.$$

   (a) Find a.
   (b) Find the cumulative distribution function $F_X(x)$.
   (c) Find $E[X]$.
   (d) Find $Var(X)$.
   (e) Let $Y = 1/X$. Find the cumulative distribution function $F_Y(y)$.
   (f) Find $E[Y]$.
   (g) Find $Var(Y)$.

2. (25 points) Let X, Y be random variables having density function

$$f_{X,Y}(x, y) = ye^{-x}\mathbf{1}_{[0<y<x]}\mathbf{1}_{[0<x<\infty]}.$$

   (a) Are X and Y independent? Very briefly explain your reasoning.
   (b) Find the marginal density of X, $f_X(x)$.
   (c) Find the marginal density of Y, $f_Y(y)$.
   (d) Find the conditional density of Y given $X = x$, $f_{X|Y}(y|x)$.
   (e) Find $E[Y \mid X = x]$.

3. (20 points) The chi-squared distribution with n degrees of freedom is derived as the distribution of the sum

$$\chi_n^2 = \sum_{i=1}^{n} Z_i^2,$$

where $Z_1, \ldots, Z_n$ are independent, identically distributed standard normal random variables (so $Z_i \sim \mathcal{N}(0,1)$). Suppose random variable $V \sim \chi_n^2$.

   (a) Derive $E[V]$.

   (b) Derive $Var(V)$.

4. (60 points) Let X_1, X_2, X_n be independent, identically distributed random variables with density

$$f_{X_i}(x) = \frac{2x}{\theta^2}\mathbf{1}_{(0,\theta)}(x).$$

   (a) Find $E[\,X_i\,]$ and $Var[\,X_i\,)$.

   (b) For $W = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, find $E[W]$ and $Var(W)$.

   (c) For what value of a would "estimator" $\hat{\theta}_W = aW$ be unbiased (i.e., have $E[\hat{\theta}_W] = \theta$)?

   (d) What is $Var(\hat{\theta}_W)$ for that value of a?

   (e) For $Y = \max(X_1, X_2, \ldots, X_n)$, find the cumulative distribution function $F_Y(y \mid \theta)$.

   (f) For $Y = \max(X_1, X_2, \ldots, X_n)$, find the probability density function $f_Y(y \mid \theta)$.

   (g) Find $E[Y]$.

   (h) Find $Var(Y)$.

   (i) For what value of b would "estimator" $\hat{\theta}_Y = bW$ be unbiased (i.e., have $E[\hat{\theta}_Y] = \theta$?

   (j) What is $Var(\hat{\theta}_Y)$ for that value of b?

   (k) Which of the unbiased estimators $\hat{\theta}_W$ and $\hat{\theta}_Y$ is more precise (i.e., has lower variance)?

5. (10 points) Suppose normally distributed random variable $Y \sim \mathcal{N}(\theta, \tau^2)$, and suppose that random variable X is conditionally normally distributed according to

$$(X|Y = y) \sim \mathcal{N}(y, y^2).$$

(a) Find $E[X]$, the unconditional expectation of X.

(b) Find $Var[X]$, the unconditional variance of X.

6. (35 points) Let $Y \sim \mathcal{B}(1, p_Y)$ and $X \sim \mathcal{B}(1, p_X)$. Further suppose

$$Pr[Y = 1, X = 1] = \theta p_Y p_X.$$

(a) Find $\rho = corr(Y, X)$.

(b) What values of θ are valid?

(c) When is $\rho = 1$?

(d) When is $\rho = 0$? Does $\rho = 0$ imply independence in this setting? Very briefly explain your reasoning.

(e) What is the minimum value of ρ as a function of p_Y and p_X.

(f) Find $Var(Y + X)$.

(g) Find $Var(Y - X)$.