Stat 513
Emerson, Winter 04
Homework #8
March 8, 2004

Supplemental written problems due Friday, March 12, 2004 at the beginning of class.

1. Let $X_i, i = 1, \ldots, n$ be i.i.d. uniform random variables with $X_i \sim \mathcal{U}(0, \theta)$, $\theta > 0$ unknown and to be estimated.

   a. Show that $X_{(n)}/\theta$ is a pivotal quantity.

   b. Derive a formula for a 100(1-α)% confidence interval for θ.

   c. Find the expected width of your confidence interval for θ.

2. Let $X_i, i = 1, \ldots, n$ be i.i.d. Bernoulli random variables with $X_i \sim \mathcal{B}(1, p)$. Suppose we observe $\sum_{i=1}^n X_i = 0$. Find a 95% upper confidence bound for p. Show that for large n, this bound is approximately $3/n$.

3. Suppose $X_i, i = 1, \ldots, n$ are independent and identically distributed random variables which, conditional upon a parameter $\theta > 0$, have the exponential distribution $\mathcal{E}(\theta)$ with density

$$f(x) = \begin{cases} \theta e^{-\theta x} & x > 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

   Consider a prior distribution for θ according to the gamma distribution $\theta \sim \Gamma(\alpha, \beta)$, with density $b(\theta) = \beta^\alpha \theta^{\alpha-1} e^{-\beta\theta}/\Gamma(\alpha)$ and mean α/β.

   a. Show that the above prior distribution is the conjugate prior for this problem.

   b. Find the posterior distribution of $\theta | (X_1, \ldots, X_n)$.

   c. Find the Bayes estimator for squared error loss, i.e., $L(\theta, d) = (\theta - d)^2$.

   d. Consider now the case of observing a single additional random variable X_{n+1} which is independent of the previous sample and distributed according to $X_{n+1}|\theta \sim \mathcal{E}(\theta)$. Using the posterior distribution found in (b) as your prior, find the posterior distribution of θ based on the observation of X_{n+1}.

   e. Compare the posterior distribution of θ derived in (d) to that obtained by using the original prior ($\theta \sim \Gamma(\alpha, \beta)$) and the total sample having $n+1$ observations $X_1, \ldots, X_n, X_{n+1}$.