Stat 513
Emerson, Win 16
Homework #1
January 6, 2016

Written solutions to the homework problems are due on Wednesday, January 13, 2015 at the beginning of class.

The homework problems are divided into "regular" and "more involved" problems. In order to facilitate multiple graders, you should hand in these categories of problems separately. That is, hand in one paper that contains only the "regular" problems, and another paper that contains only the "more involved" problems.
As noted on the syllabus, copying of homework solutions is not allowed and, when detected, will be investigated as an infraction of the academy integrity policy of the University of Washington. While it is permissible to discuss problems with other students, TAs, or the instructor in order to learn how to solve a problem, your written solutions must be prepared without directly referencing any notes or solutions derived from other students or sources found on the internet.

REGULAR PROBLEMS

1. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables having mean μ, variance $\sigma^2 > 0$, and finite third and fourth central moments γ_3 and γ_4, respectively. Find an asymptotic joint distribution for the sample mean $\overline{X}_n$ and sample variance s_n^2.

2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables having mean μ and variance $\sigma^2 > 0$, and let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables having mean ν and variance $\tau^2 > 0$. Further suppose that X_i's and Y_j's are totally independent. For notational convenience, denote $\vec{X}_n = (X_1, \ldots, X_n)$ and $\vec{Y}_n = (Y_1, \ldots, Y_n)$.

   (a) Find a method of moments estimator $\hat{\theta}_{mn} = \hat{\theta}_{mn}(\vec{X}_m, \vec{Y}_n)$ for $\theta = \mu - \nu$. Show that your estimator is unbiased.

   (b) Let $N = m + n$. Suppose further that $m/N \to \lambda \in (0, 1)$ as $N \to \infty$. Find an asymptotic distribution for $\hat{\theta}_{mn}$. That is, show that there is some $a_N \to \infty$ such that $a_N(\hat{\theta}_{mn} - \theta)$ converges in distribution to some distribution, and identify the distribution.

3. In the setting of problem 2, we are often faced with the problem that σ^2 and τ^2 are "nuisance" parameters that are unknown, but are not central to our primary statistical question. They are necessary, however, when trying to compute confidence intervals or perform hypothesis tests. In settings where

$$a_N(\hat{\theta}_N - \theta) \to_d \mathcal{N}\left(0,\ V(\sigma^2, \tau^2, \lambda)\right)$$

we most often proceed by finding a consistent estimator $\hat{V} = V(\hat{\sigma}_m^2, \hat{\tau}_n^2, \hat{\lambda})$ such that

$$a_N \left(\frac{\hat{\theta}_N - \theta}{\sqrt{\hat{V}}} \right) \to_d \mathcal{N}(0, 1).$$

Using the notation of problem 2, find the asymptotic consistency properties of $\hat{V}/V$ for each of the following commonly used estimators of V. Explicitly consider the two possibilities that $\sigma^2 = \tau^2$ and $\sigma^2 \neq \tau^2$ and explicitly consider possible values λ.

(a) *Use of a pooled variance estimate*: $\hat{\sigma}_m^2 = \hat{\tau}_n^2 = s_{P,mn}^2$ where

$$s_{P,mn}^2 = \frac{\sum_{i=1}^m (X_i - \overline{X}_m)^2 + \sum_{j=1}^n (Y_j - \overline{Y}_n)^2}{m + n - 2}.$$

(b) *Use of separate sample variance estimates*:

$$\hat{\sigma}_m^2 = \frac{1}{m-1} \sum_{i=1}^m (X_i - \overline{X}_m)^2 \qquad \hat{\tau}_n^2 = \frac{1}{n-1} \sum_{j=1}^m (Y_j - \overline{Y}_n)^2.$$

(c) *Use of an estimate derived from a combined population:* Combine the two samples into a single sample, letting $\overline{W}_{mn}$ be the sample mean

$$\overline{W}_{mn} = \frac{m\overline{X}_m + n\overline{Y}_n}{m + n}$$

$$\hat{\sigma}_m^2 = \hat{\tau}_n^2 = \frac{\sum_{i=1}^m (X_i - \overline{W}_{mn})^2 + \sum_{j=1}^n (Y_j - \overline{W}_{mn})^2}{m + n - 1}.$$

4. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables having mean $\mu > 0$ and variance $\sigma^2 > 0$, and let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables having mean $\nu > 0$ and variance $\tau^2 > 0$. Further suppose that X_i and Y_j are independent if $i \neq j$, but that the correlation between X_i and Y_j is ρ if $i = j$. For notational convenience, denote $\vec{X}_n = (X_1, \ldots, X_n)$ and $\vec{Y}_n = (Y_1, \ldots, Y_n)$. Define sample covariance

$$\begin{aligned} \hat{\rho}_n = \hat{\rho}_n(\vec{X}_n, \vec{Y}_n) &= \frac{1}{n} \sum_{i=1}^n (X_i - \overline{X}_n)(Y_i - \overline{Y}_n) \\ &= \frac{1}{n} \sum_{i=1}^n (X_i - \overline{X}_n) Y_i \\ &= \frac{1}{n} \sum_{i=1}^n X_i (Y_i - \overline{Y}_n) \\ &= \frac{1}{n} \sum_{i=1}^n (X_i Y_i) - \overline{X}_n \overline{Y}_n. \end{aligned}$$

Find an asymptotic distribution for $\hat{\rho}_n$.

5. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables having $X_i \sim \mathcal{N}(\mu, \sigma^2)$. Show that sample mean $\overline{X}_n$ and sample variance s_X^2 are independent.

MORE INVOLVED PROBLEMS

6. Consider again the setting of problem 2, but merely presume that as $N \to \infty$, we only know that $\min(m, n) \to \infty$. Prove an asymptotic distribution for $\hat{\theta}_{mn}$ in this setting.