Stat 513
Homework key 3

February 3, 2016

REGULAR PROBLEMS

1. Consider an exponential regression model relating response Y to an intercept and one predictor vectors $\overrightarrow{X}$. Suppose that the elements of $\overrightarrow{Y}$ are totally independent with

$$f_{Y_i}(y) = \lambda_i \exp(-\lambda_i y) 1_{(0,\infty)}(y)$$

with density where we have regression model

$$\log(\lambda_i) = \beta_0 + x_i\beta_1$$

Derive expressions for the efficient score vector $\overrightarrow{U}(\overrightarrow{\beta})$ and the Fisher's information matrix $\overrightarrow{J}(\overrightarrow{\beta})$.

Ans:

Let us define the design matrix X whose first column has all entries 1 and the the second column is the vector $\overrightarrow{x} = (x_1, \ldots, x_n)^T$. Let us calculate the vector $\overrightarrow{U}(\overrightarrow{\beta})$ as follows:

$$\begin{aligned}
\overrightarrow{U}(\overrightarrow{\beta}) &= \frac{\partial}{\partial \overrightarrow{\beta}} \log \prod_{i=1}^{n} f_{Y_i}(y_i) \\
&= \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \overrightarrow{\beta}} f_{Y_i}(y_i)}{f_{Y_i}(y_i)} \\
&= \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \overrightarrow{\beta}} \lambda_i \frac{\partial}{\partial \lambda_i} f_{Y_i}(y_i)}{f_{Y_i}(y_i)} \\
&= \sum_{i=1}^{n} (\frac{1}{\lambda_i} - y_i)\lambda_i \begin{bmatrix} 1 \\ x_i \end{bmatrix}
\end{aligned}$$

using the fact that $\frac{\partial}{\partial \overrightarrow{\beta}} \lambda_i = \lambda_i \begin{bmatrix} 1 \\ x_i \end{bmatrix}$. The above expression can be written as

$$\overrightarrow{U}(\overrightarrow{\beta}) = \sum_{i=1}^{n} (1 - \lambda_i y_i) \begin{bmatrix} 1 \\ x_i \end{bmatrix}$$

This gives us

$$\begin{aligned}
\overrightarrow{J}(\overrightarrow{\beta}) = &- E\left[\frac{\partial}{\partial \overrightarrow{\beta}^T} \overrightarrow{U}(\overrightarrow{\beta})\right] = \sum_{i=1}^{n} \begin{bmatrix} 1 \\ x_i \end{bmatrix} \frac{\partial}{\partial \overrightarrow{\beta}} \lambda_i y_i = \sum_{i=1}^{n} \begin{bmatrix} 1 \\ x_i \end{bmatrix} [1 \; x_i] \lambda_i E Y_i = \sum_{i=1}^{n} \begin{bmatrix} 1 \\ x_i \end{bmatrix} [1 \; x_i] \\
&= \begin{bmatrix} 1 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix}.
\end{aligned}$$

2. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. Weibull random variables having cumulative distribution function

$$F_{Y_i}(y) = (1 - \exp\{-(\lambda y)^p\}) \mathbf{1}_{(0,\infty)}(y)$$

for $\lambda \in (0, \infty), p \in (0, \infty)$. Denote the unknown parameter vector by $\overrightarrow{\theta} = (p, \lambda)$ and derive expressions for the efficient score vector $\overrightarrow{\mathcal{U}}(\overrightarrow{\theta})$ and the Fisher's information matrix $\mathbf{J}(\overrightarrow{\theta})$.

Ans:

We first obtain the density $f_{Y_i}(y)$ by differentiating the cdf,

$$f_{Y_i}(y) = \frac{d}{dy} F_{Y_i}(y) = p\lambda(\lambda y)^{p-1} e^{-(\lambda y)^p} \mathbf{1}_{(0,\infty)}(y).$$

The efficient score is obtained by taking the gradient of the log likelihood function,

$$\overrightarrow{\mathcal{U}}(\overrightarrow{\theta}) = \frac{\partial}{\partial \overrightarrow{\theta}} \log f_{Y_i}(y) = \left(\frac{\partial}{\partial p} \log f_{Y_i}(y), \frac{\partial}{\partial \lambda} \log f_{Y_i(y)}\right)^T = \left(\frac{1}{p} - ((\lambda y)^p - 1)\log(\lambda y), \quad -\frac{p}{\lambda}((\lambda y)^p - 1)\right)^T.$$

It can be verified that for the 2-parameter Weibull distribution in question, we have that Y follows a regular parametric model. We can thus calculate the information matrix based on the negative 2nd-order derivatives of the log likelihood,

$$-\frac{\partial}{\partial \overrightarrow{\theta}} \overrightarrow{\mathcal{U}}(\overrightarrow{\theta}) = -\frac{\partial^2}{\partial \overrightarrow{\theta}^2} \log f_{Y_i}(y) = \begin{pmatrix} \frac{1}{p^2} + (y\lambda)^p \log^2(y\lambda) & \frac{p}{\lambda}(\log(y\lambda)(y\lambda)^p + (y\lambda)^p - 1) \\ \frac{p}{\lambda}(\log(y\lambda)(y\lambda)^p + (y\lambda)^p - 1) & \frac{p}{\lambda^2}(p(y\lambda)^p - (y\lambda)^p + 1) \end{pmatrix}.$$

The information is calculated as

$$\mathbf{J}(\overrightarrow{\theta}) = E\begin{pmatrix} \frac{1}{p^2} + (y\lambda)^p \log^2(y\lambda) & \frac{p}{\lambda}(\log(y\lambda)(y\lambda)^p + (y\lambda)^p - 1) \\ \frac{p}{\lambda}(\log(y\lambda)(y\lambda)^p + (y\lambda)^p - 1) & \frac{p}{\lambda^2}(p(y\lambda)^p - (y\lambda)^p + 1) \end{pmatrix},$$

where the requisite integrations do not have a closed form. The expectations could be simplified using the substitution $t = (y\lambda)^p$. Further simplifications can be made if one recognizes the digamma and trigamma functions, resulting in a final information matrix of

$$\mathbf{J}(\overrightarrow{\theta}) = \begin{pmatrix} \frac{1}{p^2}\left((1-\gamma)^2 + \frac{\pi^2}{6}\right) & \frac{1-\gamma}{\lambda} \\ \frac{1-\gamma}{\lambda} & \frac{p^2}{\lambda^2} \end{pmatrix},$$

where $\gamma = 0.5772...$ is the Euler-Mascheroni constant.