Stat 513
Emerson, Win 16
Homework #4
January 29, 2016

Written solutions to the homework problems are due on Wednesday, February 3, 2016 at the beginning of class.

The homework problems are divided into "regular" and "more involved" problems. In order to facilitate multiple graders, you should hand in these categories of problems separately. That is, hand in one paper that contains only the "regular" problems, and another paper that contains only the "more involved" problems.
As noted on the syllabus, copying of homework solutions is not allowed and, when detected, will be investigated as an infraction of the academy integrity policy of the University of Washington. While it is permissible to discuss problems with other students, TAs, or the instructor in order to learn how to solve a problem, your written solutions must be prepared without directly referencing any notes or solutions derived from other students or sources found on the internet.

REGULAR PROBLEMS

1. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. lognormal random variables with $Y_i \sim \mathcal{LN}(\mu, \sigma^2)$. Denote the unknown parameter vector by $\vec{\theta} = (\mu, \sigma^2)$.
   (a) Derive the asymptotic distribution for the maximum likelihood estimate $\hat{\eta}$ of $\eta = E[Y_i]$.
   (b) Derive the asymptotic distribution for the nonparametric estimate $\tilde{\eta}$ of η.
   (c) Find the asymptotic relative efficiency
   $$e(\tilde{\eta}, \hat{\eta}) = \frac{Var(\hat{\eta})}{Var(\tilde{\eta})}.$$
2. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. exponential random variables with $Y_i \sim \mathcal{E}(\lambda)$ with hazard λ. Further define random variables $W_i = \mathbf{1}_{[Y_i \le c]}$ for scalar c.
   (a) Derive the asymptotic distribution for the maximum likelihood estimate $\hat{\lambda}$ of λ based on $\vec{Y}$.
   (b) Derive the asymptotic distribution for the maximum likelihood estimate $\tilde{\lambda}$ of λ based on $\vec{W}$.
   (c) Evaluate the asymptotic relative efficiency
   $$e(\tilde{\lambda}, \hat{\lambda}) = \frac{Var(\hat{\lambda})}{Var(\tilde{\lambda})}$$

as a function of c. Comment on the efficiency loss due to dichotomization of the data.

MORE INVOLVED PROBLEMS

3. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. continuous random variables with cdf $F_Y(y;\theta)$ and density $f_Y(y;\theta)$. Let $\eta = F^{-1}(p)$ be the p-th quantile of the distribution of Y, and suppose that $f(\eta) > 0$. Then, the sample quantile $\tilde{\eta}$ computed using $\vec{Y}_n = (Y_1, \ldots, n)$ can be shown to have asymptotic distribution

$$\sqrt{n}(\tilde{\eta} - \eta) \to_d \mathcal{N}\left(0, \frac{p(1-p)}{f^2(\eta \mid \theta)}\right).$$

Suppose we are interested in estimating medians of the distribution (so $p = 0.5$). For each of the following continuous distributions, find the asymptotic relative efficiency of the sample median $\tilde{\eta}$ compared to the MLE $\hat{\eta}$ of η.

(a) Normal distribution: $Y_i \sim \mathcal{N}(\mu, \sigma^2)$.

(b) Exponential distribution: $Y_i \sim \mathcal{E}(\lambda)$.

(c) Y_i has density

$$f_Y(y \mid \sigma^2) = \frac{y}{\sigma^2} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} \mathbf{1}_{(0,\infty)}(y).$$

(d) Uniform distribution: $Y_i \sim \mathcal{U}(0, \theta)$. (Find the ratio of variances for the approximate distribution based on the asymptotic distributions.)