Stat 513
Emerson, Win 16
Homework #5
February 6, 2016

Written solutions to the homework problems are due on Tuesday, February 9, 2016 at the beginning of class.

Note that this homework is meant to help you review for the midterm. To that end, each problem is thought to be appropriate for inclusion on an in-class, closed book exam. Hence, you may get the most out of this homework if you first try to complete it in, say, a two hour period after studying the previous material. (There are more problems here than I would expect you to complete in a 50 minute exam.)

As noted on the syllabus, copying of homework solutions is not allowed and, when detected, will be investigated as an infraction of the academy integrity policy of the University of Washington. While it is permissible to discuss problems with other students, TAs, or the instructor in order to learn how to solve a problem, your written solutions must be prepared without directly referencing any notes or solutions derived from other students or sources found on the internet.

REGULAR PROBLEMS

1. Suppose $Y_1, Y_2, \dots, Y_n$ are i.i.d. Bernoulli random variables with $Y_i \sim \mathcal{B}(1,p)$, $p \in (0,1)$. Denote the target of inference by $\theta = Var(Y_i)$.
   (a) What is the Cramér-Rao lower bound for the variance of an unbiased estimator of θ.
   (b) Can you find an estimator that achieves that lower bound? If so, derive that estimator. If not, explain why not.
   (c) Find the maximum likelihood estimate $\hat{\theta}$ of θ.
   (d) Derive an asymptotic distribution for $\hat{\theta}$ in the case that $p \neq \frac{1}{2}$.
   (e) Find the asymptotic relative efficiency of $\hat{\theta}$ to the unbiased nonparametric estimator s^2, the sample variance, when $p \neq \frac{1}{2}$.
   (f) Why is the asymptotic distribution for $\hat{\theta}$ derived in part 4 not useful when $p = \frac{1}{2}$? Derive an asymptotic distribution that could be used when $p = \frac{1}{2}$.
2. Suppose $Y_1, Y_2, \dots, Y_n$ are i.i.d. normal random variables with $Y_i \sim \mathcal{N}(\theta, \theta)$ with $\theta \in (0, \infty)$.
   (a) What is the Cramér-Rao lower bound for the variance of an unbiased estimator of θ.

(b) For what function $g(\theta)$ can you derive a best regular unbiased estimator (BRUE)? Justify your answer.

(c) Find the maximum likelihood estimate $\hat{\theta}$ of θ.

(d) Derive an asymptotic distribution for $\hat{\theta}$.

3. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. random variables which for some $\theta \in (2, \infty)$ have density

$$f_Y(y \mid \theta) = \theta y^{-(\theta+1)} \mathbf{1}_{(1,\infty)}(y).$$

(a) Find the maximum likelihood estimate $\hat{\theta}$ of θ.

(b) Prove that $\hat{\theta}$ is biased.

(c) Find the asymptotic distribution of the maximum likelihood estimate for $g(\theta) = Var(Y)$.

4. Consider a "dose-response" regression in which we presume that the mean "response" Y_i is related to "dose" X_i by

$$(Y_i \mid X_i = x_i) = \beta_0 + \beta_1 X_i + \epsilon_i,$$

where we presume the ϵ_i's are i.i.d. with $\epsilon_i \sim (0, \sigma^2)$. We are interested in estimating the dose θ_{50} such that $E[Y \mid X = \theta_{50}] = 50$.

(a) Derive expressions for the best linear unbiased estimator $\hat{\vec{\beta}}$ of $\vec{\beta}$.

(b) What is the asymptotic distribution of $\hat{\vec{\beta}}$? Be sure to provide the assumptions under which you derived the asymptotic distribution.

(c) Derive an expression for an estimator $\hat{\theta}_{50} = g(\hat{\vec{\beta}})$, along with its asymptotic distribution.

MORE INVOLVED PROBLEMS

5. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. random variables with the double exponential distribution having density for $\theta > 0$

$$f_Y(y \mid \theta) = \frac{1}{2}\theta e^{-\theta|y|}.$$

(a) Find a method of moments estimate $\tilde{\theta}$ for θ and derive its asymptotic distribution.

(b) Find a maximum likelihood estimate $\hat{\theta}$ for θ.

(c) Derive the asymptotic distribution of $\hat{\theta}$. (Hint: Is this a regular problem? What can you use if it is not?)

(d) What is the asymptotic relative efficiency of the MME $\tilde{\theta}$ compared to MLE $\hat{\theta}$.