Stat 513
Emerson, Win 16
Homework #6
February 20, 2016

Written solutions to the homework problems are due on Friday, February 26, 2016 at the beginning of class.

As noted on the syllabus, copying of homework solutions is not allowed and, when detected, will be investigated as an infraction of the academy integrity policy of the University of Washington. While it is permissible to discuss problems with other students, TAs, or the instructor in order to learn how to solve a problem, your written solutions must be prepared without directly referencing any notes or solutions derived from other students or sources found on the internet.

REGULAR PROBLEMS

1. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. Bernoulli random variables with $Y_i \sim \mathcal{B}(1,p)$, $p \in (0,1)$. Denote the target of inference by $\theta = Var(Y_i)$.
   (a) Derive a uniform minimum variance unbiased estimator (UMVUE) $\tilde{\theta}$ of θ.
   (b) Find the variance of $\tilde{\theta}$. Does it meet the Cramér-Rao lower bound for an unbiased estimator of θ?
   (c) Derive an exact probability distribution for $\tilde{\theta}$.
   (d) Derive an asymptotic distribution for $\tilde{\theta}$.
2. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. normal random variables with $Y_i \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu \in (-\infty, \infty)$ and $\sigma^2 \in (0, \infty)$. Denote the target of inference by $\theta = Var(Y_i) = \sigma^2$.
   (a) Derive a uniform minimum variance unbiased estimator (UMVUE) $\tilde{\theta}$ of θ.
   (b) Find the variance of $\tilde{\theta}$. Does it meet the Cramér-Rao lower bound for an unbiased estimator of θ?
   (c) Derive an exact probability distribution for $\tilde{\theta}$.
   (d) Derive an asymptotic distribution for $\tilde{\theta}$.
3. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. normal random variables with $Y_i \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu \in (-\infty, \infty)$ and $\sigma^2 \in (0, \infty)$. Denote the target of inference by $\theta = \mu^2$.
   (a) Derive a uniform minimum variance unbiased estimator (UMVUE) $\tilde{\theta}$ of θ.
   (b) Find the variance of $\tilde{\theta}$. Does it meet the Cramér-Rao lower bound for an unbiased estimator of θ?
   (c) Derive an exact probability distribution for $\tilde{\theta}$.

(d) Derive an asymptotic distribution for $\tilde{\theta}$.

4. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. exponential random variables with $Y_i \sim \mathcal{E}(\mu)$ with mean $\mu \in (0, \infty)$. Denote the target of inference by $\theta = Var(Y_i) = \mu^2$.

   (a) Derive a uniform minimum variance unbiased estimator (UMVUE) $\tilde{\theta}$ of θ.

   (b) Find the variance of $\tilde{\theta}$. Does it meet the Cramér-Rao lower bound for an unbiased estimator of θ?

   (c) Derive an exact probability distribution for $\tilde{\theta}$.

   (d) Derive an asymptotic distribution for $\tilde{\theta}$.

5. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. uniform random variables with $Y_i \sim \mathcal{U}(0, \mu)$ with $\mu \in (0, \infty)$. Denote the target of inference by $\theta = Var(Y_i)$.

   (a) Derive a uniform minimum variance unbiased estimator (UMVUE) $\tilde{\theta}$ of θ.

   (b) Find the variance of $\tilde{\theta}$. Does it meet the Cramér-Rao lower bound for an unbiased estimator of θ?

   (c) Derive an exact probability distribution for $\tilde{\theta}$.

   (d) Derive an asymptotic distribution for $\tilde{\theta}$.

6. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. exponential random variables with $Y_i \sim \mathcal{E}(\lambda)$ with hazard $\lambda \in (0, \infty)$. Denote the target of inference by $\theta = Pr(Y_1 > 1)$.

   (a) Derive a uniform minimum variance unbiased estimator (UMVUE) $\tilde{\theta}$ of θ.

   (b) Derive an expression for the variance of $\tilde{\theta}$. Does it meet the Cramér-Rao lower bound for an unbiased estimator of θ?

   (c) Derive an exact probability distribution for $\tilde{\theta}$.

   (d) Derive an asymptotic distribution for $\tilde{\theta}$.

MORE INVOLVED PROBLEMS

7. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. Bernoulli random variables with $Y_i \sim \mathcal{B}(1, p)$, $p \in (0, 1)$. Denote the target of inference by $p = E(Y_i)$. Suppose further that the data are sample sequentially with the following (fairly stupid) group sequential stopping rule with potential analyses after $n_1 = 1$ and $n_2 = 11$ observations have been observed:

   - Y_1 is sampled.
   - If $Y_1 = 1$, we stop sampling at $n_1 = 1$ and define $(M, S) = (1, Y_1)$.
   - If $Y_1 = 0$, we sample $Y_2, \ldots, Y_{11}$ and then stop sampling at $n_2 = 11$. We define $(M, S) = (2, \sum_{i=1}^{11} Y_i)$.

Note that under this scheme, M is our (random) stopping time, and $S = \sum_{i=1}^{n_M} Y_1$. In group sequential terminology, the "continuation set" at the first analysis is $\mathcal{C}_1 = \{0\}$.

(a) Derive the exact distribution for statistic (M, S).

(b) Show that the exact distribution is a curved exponential family distribution.

(c) Find a minimal sufficient statistic for p.

(d) Show that minimal sufficient statistic for p is a complete sufficient statistic.

(e) Find a uniform minimum variance unbiased estimator (UMVUE) $\tilde{p}$ of p.

(f) Find the maximum likelihood estimator (MLE) $\hat{p}$ of p, and show that it is a biased estimator.

(g) Compare the MSE of the MLE $\hat{p}$ to the MSE of the UMVUE $\tilde{p}$.

(h) Derive a "bias adjusted estimator" $\check{p}$ such that

$$E\left[\frac{S}{n_M} \,|\, p = \check{p}\right] = \hat{p}.$$