Stat 513
Emerson, Win 16
Retake: Midterm Examination
February 19, 2016

**Instructions for Retake:**

**I am allowing an option for students to complete or correct their answers to the midterm examination. This "retake option" as a take-home exam is subject to the following conditions:**

- **By submitting a corrected exam, you can earn up to half the total number of points taken off from your solution as submitted during the in-class portion of the exam.**
- **There is no limit to the amount of time you can spend on the take-home portion of the exam, except that any corrections/additions must be handed in by the time that the clock in the classroom registers 10:35 am on Monday, February 22, 2016. There are no possible exceptions to this rule. (If you anticipate being hit by a meteor on the way to class, I would encourage your making other arrangements to get your exam to class on time.)**
- **You may work on the problems using the texts Casella and Berger, Dudewicz and Mishra, lecture notes that you took in class, or notes obtained from the class web pages. You may not use any other book or notes from any other class. You may not talk about the problems with anyone else prior to turning in your corrections/additions. This includes me and the TAs. We will not answer any questions either in person or via email about any matters regarding this examination.**
- **In order for the take-home portion to be accepted, it must be accompanied by the following (truthful) pledge written out and signed by you:**
- **"On my honor, I have neither given nor received any unauthorized aid on this take-home examination."**
- **In the event that you have inadvertently violated this pledge (e.g., if you overheard a discussion of the problems by someone who is not turning in corrections/additions), you should not sign the pledge and instead discuss the situation with me. Again, there are no exceptions to this policy, as violations of a signed pledge will be treated as an infraction of the Academic Integrity policy of the University of Washington.**

**Note:** The typo in problem 2c has been corrected. You should work the problem as stated on this exam, which stipulates that the data is heteroscedastic.

1. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. normal random variables with $Y_i \sim \mathcal{N}(\theta, \theta^2)$ with $\theta \in (-\infty, \infty)$.

   (a) What is the Cramér-Rao lower bound for the variance of an unbiased estimator of $\theta$.

   (b) Is there a function $g(\theta)$ for which you can derive a best regular unbiased estimator (BRUE)? Justify your answer.

   (c) Find the maximum likelihood estimate $\hat{\theta}$ of $\theta$.

   (d) Derive an asymptotic distribution for $\hat{\theta}$.

   (e) Find the asymptotic relative efficiency of the MLE $\hat{\theta}$ compared to the sample mean $\overline{Y}$.

   (f) Find a MME estimator $\tilde{\theta}$ based on the sample variance, and derive its asymptotic distribution. What is the asymptotic relative efficiency of the MLE $\hat{\theta}$ compared to the MME $\tilde{\theta}$?

2. Consider a simple linear regression in which we presume that the mean of $Y_i$ is related to known predictor $X_i$ by

$$(Y_i \mid X_i = x_i) = \beta_0 + \beta_1 X_i + \epsilon_i,$$

   where we presume the $\epsilon_i$'s are independent with $\epsilon_i \sim (0, \sigma_i^2)$.

   (a) Derive expressions for the ordinary least squares estimator $\hat{\vec{\beta}}_{OLS}$ of $\vec{\beta}$, and provide expressions for its mean and variance, making clear any special notation you use.

   (b) Derive expressions for the best linear unbiased estimator $\hat{\vec{\beta}}_B$ of $\vec{\beta}$, and provide expressions for its mean and variance, again making clear any special notation you use.

   (c) Suppose that $X_i = x_i > 0$ for all $i$ and $\sigma_i^2 = \theta^2 x_i$. Write down an expression for an unbiased estimator of $\theta^2$.

3. Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. random variables with the double exponential distribution having density for $\mu \in (-\infty, \infty)$

$$f_Y(y \mid \mu) = \frac{1}{2} e^{-|y-\mu|}.$$

   (a) Is this a regular probability model? Explain your answer.

   (b) Find a method of moments estimate $\tilde{\mu}$ for $\mu$ and derive its asymptotic distribution. (Hint: What is the distribution of a random variable $W_i = Y_i - \mu$.)

   (c) Find the asymptotic distribution of maximum likelihood estimate $\hat{\mu}$ (the sample median) for $\mu$. (You do not need to prove the form of the MLE.)

   (d) What is the asymptotic relative efficiency of the MME $\tilde{\mu}$ compared to MLE $\hat{\mu}$.

4. Suppose $W_1, W_2, \ldots, W_n$ are i.i.d. multinomial random variables with $W_i \sim \mathcal{M}\sqcap\updownarrow\sqcup(1, \vec{\theta})$, such that $W_i \in \{1, 2, 3\}$ with $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ satisfying $\theta_k = Pr(W_i = k)$, $\theta_k \in (0, 1)$, and $\sum_{k=1}^{3} \theta_k = 1$. For notational convenience, we can define random vector $\vec{Y}_i$ as

$$\vec{Y}_i = \begin{pmatrix} Y_{i1} = \mathbf{1}_{[W_i=1]} \\ Y_{i2} = \mathbf{1}_{[W_i=2]} \\ Y_{i3} = \mathbf{1}_{[W_i=3]} \end{pmatrix}$$

We also note that owing to the constraints on the multinomial random variables, we will find it easiest to reparameterize our problem such that $\theta_3 = 1 - \theta_1 - \theta_2$ and $Y_{i3} = 1 - Y_{i1} - Y_{i2}$. Hence, we have density for $\vec{y}$ having exactly one nonzero element equal to 1

$$f_{\vec{Y}_i}(\vec{y} \,|\, \vec{\theta}) = \theta_1^{y_1} \theta_2^{y_2} \theta_3^{y_3} = \theta_1^{y_1} \theta_2^{y_2} (1 - \theta_1 - \theta_2)^{1 - y_1 - y_2}.$$

(a) Write down the likelihood and log likelihood of $\vec{\theta}$ based on observations $\vec{Y}_1, \vec{Y}_2, \ldots$.

(b) Find the maximum likelihood estimate $\hat{\vec{\theta}}$ of $\vec{\theta}$, and derive its mean and variance. Is it unbiased?

(c) What is the Cramér-Rao lower bound for variance-covariance matrix of an unbiased estimator of $\vec{\theta}$?

(d) Derive an asymptotic distribution for $\hat{\vec{\theta}}$.

(e) Suppose we are interested in inference about $g(\vec{\theta}) = \theta_1 - (\theta_2 + \theta_3)$. Find the asymptotic distribution for the MLE of $g(\vec{\theta})$.