Stat 582
Emerson, Win 12
Homework #1
January 6, 2012

Written problems to be handed in Friday, January 13, 2012.

1. Let $T_n = \sum_{i=1}^n X_i/n$, where $X_1, X_2, \dots$ are independent random variables having probability distribution given by
$$Pr(X_i = -i) = Pr(X_i = i) = \frac{1}{2}.$$
   a. Show that the sequence of statistics $\{T_n\}_{n=1}^\infty$ is asymptotically unbiased for some θ (and identify θ), where a sequence of statistics $\{T_n\}_{n=1}^\infty$ is called *asymptotically unbiased* for θ if
$$\lim_{n\to\infty} E(T_n - \theta) = 0.$$
   b. Show that the sequence of statistics $\{T_n\}_{n=1}^\infty$ is not consistent for θ.

2. Chebyshev's inequality states that for a random variable X having finite variance σ^2 and for any $\epsilon > 0$
$$Pr\left(|X - E(X)| \geq \epsilon\sigma\right) \leq \frac{1}{\epsilon^2}.$$
   a. Show that the upper bound on the tail probabilities can be extremely conservative by finding a distribution for which the true bound is 0 for all $\epsilon > 1$.
   b. Show that the upper bound is not always conservative by showing that for each choice of $\epsilon > 1$ there is some random variable X_ϵ that has the tail probabilities exactly equal to the Chebyshev bound.

3. Let sequence of statistics $\{T_n\}_{n=1}^\infty$ be asymptotically unbiased for θ. Furthermore, suppose that
$$\lim_{n\to\infty} Var(T_n) \to 0.$$
Show that $\{T_n\}_{n=1}^\infty$ is consistent for θ.

4. Let $X_1, X_2, \dots$ be independent random variables for which $X_n \sim \chi^2_n$, a chi squared distribution with n degrees of freedom. Let sequence of statistics $\{T_n\}_{n=1}^\infty$ be defined by $T_n = X_n/n$. Show that $\{T_n\}_{n=1}^\infty$ is consistent for some θ (and identify θ).

5. Suppose sequence of random variables $\{X_n\}_{n=1}^\infty$ converge almost surely to random variable X. Suppose further that function g is continuous. Rigorously prove that $g(X_n) \to_{as} g(X)$.

6. Provide counterexamples demonstrating the falseness of each of the following statements.
   a. $X_n \to_d X$ implies $X_n \to_p X$.
   b. $X_n \to_d X$ implies $X_n \to_{as} X$.
   c. $X_n \to_p X$ implies $X_n \to_{as} X$.
   d. $X_n \to_p X$ implies $X_n \to_r X$ for $r = 1$.
   e. $X_n \to_r X$ implies $X_n \to_{r'} X$ for $r' > r$.

7. Let $X_1, X_2, \dots$ be random variables having means $EX_i = \mu_i < \infty$, variance $Var(X_i) = \sigma^2 < \infty$, with $corr(X_i, X_j) = 0$ for $i \neq j$ (note that the variables need not be independent, just uncorrelated). Define $\overline{X}_n = \sum_{i=1}^n X_i/n$ and $\theta_n = \sum_{i=1}^n \mu_i/n$. Show that the sequence of statistics $\{T_n\}_{n=1}^\infty$ with $T_n = \overline{X}_n - \theta_n$ satisfies $T_n \to_p 0$.