Written problems to be handed in Friday, February 3, 2012.

1. Prove: If sequence of random variables $X_n \rightarrow_d a$, where a is constant, then $X_n \rightarrow_p a$.
2. Prove: If sequence of random variables $X_n \rightarrow_d X$, then $Var(X) \leq \liminf_{n\rightarrow\infty} Var(X_n)$.
3. Prove the delta method: Suppose $a_n \rightarrow \infty$ as $n \rightarrow \infty$ and X_n is a sequence of random variables. Further suppose $g : \mathcal{R}^1 : \mathcal{R}^1$ is a continuous function that is differentiable at θ. Then

$$a_n\left(X_n - \theta\right) \rightarrow_d X \qquad \Rightarrow \qquad a_n\left(g(X_n) - g(\theta)\right) \rightarrow_d g'(\theta)X.$$

4. Consider known sequence of predictors $\{x_i\}_{i=1}^{\infty}$, independent identically distributed random variables $\{\epsilon_i\}_{i=1}^{\infty}$ with $\epsilon_i \sim (0, \sigma^2 < \infty)$, and derived random variables $Y_i = \alpha + \beta x_i + \epsilon_i$. Let $\hat{\theta} = (\hat{\alpha}, \hat{\beta})^T$ be the ordinary least squares estimates of $\theta = (\alpha, \beta)^T$.
   a. Show that for suitable restrictions on the x_i's (and make clear what those restrictions are)

$$V_n = \begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix}$$

   we have

$$V_n^{1/2}(\hat{\theta} - \theta) \rightarrow_d \mathcal{N}_2(0, \sigma^2 I_2),$$

   where I_2 is the identity matrix. (Hint: consider the Cramér-Wold device and the Lindeberg-Feller central limit theorem.)
   b. Show that $\hat{\sigma}_n^2 = \sum_{i=1}^n (Y_i - \hat{\alpha} - \hat{\beta}x_i)^2/(n-2)$ is a consistent estimator for σ^2.
   c. Show that the normal theory based inference for ordinary least squares regression estimates (i.e., statistical inference based on the t and F distributions) in this homoscedastic setting is thus asymptotically valid for any error distribution with a finite variance.
5. Consider identically distributed random variables

$$X_i \sim (\mu, \sigma^2 \in (0, \infty)) \quad \text{having correlation } corr(X_i, X_j) = \rho \text{ with } 0 < \rho < 1 \text{ for } i \neq j.$$

Prove or disprove the following statement about the sample mean: $\overline{X}_n \rightarrow_p \mu$ as $n \rightarrow \infty$. (If you want, you can extend your answer to the three additional special cases when $\rho < 1$, $\rho = 0$, and $\rho = 1$.