Stat 582
Emerson, Win 12
Homework #4
February 6, 2012

Written problems to be handed in Monday, February 13, 2012.

1. For the following estimators, find consistent estimators of the mean based on the sample median, and provide their asymptotic distributions. In each case, find the asymptotic relative efficiency of median based estimator to the sample mean.
   a. The normal distribution $X \sim \mathcal{N}(\mu, \sigma^2)$.
   b. The exponential distribution $f(x) = \lambda e^{-x\lambda} 1_{(0,\infty)}(x)$.
   c. The Laplace distribution $f(x) = \lambda e^{-|x-\mu|\lambda}/2$.
   d. The uniform distribution $\mathcal{U}(0, \theta)$.
   e. The lognormal distribution X such that $\log X \sim \mathcal{N}(\mu, \sigma^2)$.
2. Find an approximate distribution for the interquartile range of a continuous random variable, where the interquartile range is defined as the difference between the sample 75th and 25th percentiles. Provide explicit expressions for the cases of a normal, exponential, and uniform random variable.
3. Consider a Bayesian analysis that assumes $X_1, X_2, \ldots$ are independently distributed according to $X_i|\mu \sim \mathcal{N}(\mu, \sigma^2)$ with σ^2 known and $\mu \sim \mathcal{N}(\zeta, \tau^2)$.
   a. Find an expression for the $Pr(\mu \leq \mu_0 | X_1, \ldots, X_n)$ as would be derived for such an analysis.
   b. Suppose that the data are not truly normally distributed, but do have conditional mean μ and variance σ^2. Show that the expression you derived in part a is a valid approximation for a posterior probability and describe the sense in which it is valid, i.e., how might it differ from the posterior distribution we would have obtained if we knew the true distribution of the data.
4. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $X_i \sim \mathcal{N}(\mu, 1)$. For some $0 < \epsilon < 1$
$$T_n = \begin{cases} \overline{X}_n & \text{if } |\overline{X}_n| \geq n^{-1/4} \\ \\ \epsilon \overline{X}_n & \text{if } |\overline{X}_n| < n^{-1/4}. \end{cases}$$
   a. What is the distribution of T_n? Is it unbiased for μ? Is it asymptotically unbiased?
   b. Show that T_n is consistent for μ.
   c. Find the asymptotic distribution of $\sqrt{n}(T_n - \mu)$ as a function of μ.
   d. Comment on the variance of the limiting distribution derived in part a relative to the Cramér-Rao lower bound specified for finite samples for T_n.