Stat 582
Emerson, Win 12
Homework #6
March 5, 2012

Written problems to be handed in Friday, March 9, 2012.

1. Suppose $X_1, X_2, \ldots$ are independent Poisson random variables with $X_i \sim \mathcal{P}(\lambda)$. Verify that this is a "regular" problem for inference about $\lambda$, including that the score statistic is consistent for testing $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$.

2. Suppose $X_1, X_2, \ldots$ are independent Poisson random variables with $X_i \sim \mathcal{P}(\lambda t_i)$ for known scalars $t_i$ and $Y_1, Y_2, \ldots$ are independent Poisson random variables with $Y_i \sim \mathcal{P}(\omega u_i)$ for known scalars $u_i$. We are interested in testing the rate ratio $\theta = \lambda/\omega$ using samples $X_i$ for $i = 1, \ldots, n$ and $Y_j$ for $j = 1, \ldots, m$ where $m/n \to r > 0$ as $n \to \infty$.

   a. Show that likelihood inference about $\theta$ can be based on the statistics $S_n = \sum_{i=1}^{n} X_i$ and $T_m = \sum_{j=1}^{m} Y_j$.

   b. Derive general expressions for the score, Wald, and likelihood ratio statistics for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.

   c. Provide expressions for the score, Wald, and likelihood ratio statistics (and their critical values) for a level $\alpha$ test of $H_0$ for the special case of $\theta_0 = 1$.

   d. Suppose we observe $S_{100} = 335$ for $\sum_{i=1}^{100} t_i = 52$ and $T_{50} = 160$ for $\sum_{i=1}^{50} u_i = 28$. Derive the p value for testing null hypotheses with the score, Wald, and likelihood ratio statisics for the cases that $\theta_0 = 1$, $\theta_0 = 1.3$, and $\theta_0 = 0.8$. (Be sure to provide efficient likelihood estimates of $\theta$ overall and under the null hypothesis in each case).

3. Let $X_1, X_2, \ldots$ be independent random variables distributed according to a mixture of normals in which $X_i \sim f_X(x)$ where density $f_X$ is derived from a mixture having probability $1-p$ of a $\mathcal{N}(\mu, \sigma^2)$ distribution and probability $p$ of a $\mathcal{N}(\mu + \delta, \sigma^2)$ distribution, where $\sigma^2$ is known and $\mu$ and $\delta$ are unknown.

   a. Suppose $p$ is known. Derive the score, Wald, and likelihood ratio statistics for testing $H_0 : \delta = 0$ versus $H_1 : \delta \neq 0$. What are the asymptotic distributions of these statistics? Justify your answer.

   b. Suppose $p$ is unknown. Derive the score, Wald, and likelihood ratio statistics for testing $H_0 : \delta = 0$ versus $H_1 : \delta \neq 0$. What are the asymptotic distributions of these statistics? Justify your answer.

4. Consider the censored data setting in which independent, identically distributed random variables $T_i \sim F$ for $i = 1, \ldots, n$ are subject to censoring by known $C_i$ which are independent of the $T_i$'s. Hence, we observe only $Y_i = \min(T_i, C_i)$ and $\delta_i = 1_{[T_i = Y_i]}$. Find the nonparametric maximum likelihood estimator of $F$. Does the theory about regular asymptotic theory apply? (Hint: Argue that the likelihood of the data will be maximized by a discrete distribution function, and consider estimates of the hazard function

$$\lambda(t) = \lim_{\epsilon \to 0} \frac{P(T > t + \epsilon \,|\, T \geq t)}{\epsilon},$$

using the fact that the $T_i$'s are independent of the $C_i$'s.)